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INTRODUCTION
TO
HIGHER GEOMETRY

BY

WILLIAM C. GRAUSTEIN, Ph.D.

ASSOCIATE PROFESSOR OF MATHEMATICS
IN HARVARD UNIVERSITY

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PREFACE

The principal aim of this book is to initiate the student in the basic ideas and methods of Higher Geometry and to furnish him with an adequate background for further geometrical studies. It aspires, not to cover a large range of topics, but to develop a few fields thoroughly, with special emphasis on fundamentals which are common to all geometry.

Of the groups of geometrical transformations, the most important are the projective group and the group of circular transformations, and it is the geometries associated with these groups which are selected for intensive study.

Since the book is intended as an introduction to the subject, it is based on the student's previous training. Though the lack of a logical foundation makes the achievement of absolute rigor a difficult, if not an impossible, task, it may not be used as an excuse for treating rigor too lightly. If the student is to come to an appreciation of the need of rigor in geometry, as well as in other mathematical fields, he must be trained in accurate geometric thought. Accordingly, considerable emphasis is laid on this point. In particular, the phrase "in general" is used only sparingly. This phrase, when properly defined, is on occasion advantageous. But frequently the cases excluded by it are as interesting as the general case itself. And, what is of greater moment, placing it at the disposal of the student at an early stage subjects him to the temptation to use it as a cover for slackness.

A well balanced book on geometry should employ both synthetic and analytic methods. Of the two, the analytic methods are the more indispensable; without them, in particular, without the simple introduction of imaginary elements which they afford, the handicap is severe. Moreover, synthetic methods are

A final word, without which this preface would be woefully incomplete. For what the book contains I am responsible. But whatever success it may achieve is due to the course at Harvard University known as Mathematics 3, or better, to those of my past and present colleagues who inaugurated and developed it.

WILLIAM C. GRAUSTEIN

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INTRODUCTION TO HIGHER GEOMETRY

CHAPTER I

LINEAR EQUATIONS AND LINEAR DEPENDENCE

1. Introduction. Matrices. In the analytic treatment of geometry there are certain algebraic tools which are indispensable. The most important among them are determinants, the theory of linear equations, and the closely related theory of linear dependence.

It is assumed that the reader has a working knowledge of determinants, and that he is familiar with the application of them to the solution of n linear equations in n unknowns which goes by the name of *Cramer's rule* and reads, when $n = 3$, as follows.*

The three linear equations in three unknowns,

$$(1)\qquad \begin{aligned} a_1x + b_1y + c_1z &= k_1, \\ a_2x + b_2y + c_2z &= k_2, \\ a_3x + b_3y + c_3z &= k_3, \end{aligned}$$

when the determinant of the coefficients of the unknowns is not zero:

$$|a\,b\,c| \neq 0,$$

have one and only one simultaneous solution, namely,

$$x = \frac{|k\,b\,c|}{|a\,b\,c|}, \qquad y = \frac{|a\,k\,c|}{|a\,b\,c|}, \qquad z = \frac{|a\,b\,k|}{|a\,b\,c|},$$

where $|k\,b\,c|$, *for example, is the determinant obtained from* $|a\,b\,c|$ *by replacing each* a *by the corresponding* k.

To help us in pursuing further the study of linear equations, we need a new concept, that of a *matrix*. In the system of equations (1), the number of equations is equal to the number of unknowns, and hence the coefficients of the unknowns form a square array. This is, however, a special case. In general, the number of equations is different from the number of unknowns and the array of coefficients is rectangular.

* A knowledge of §§ 1–6, 8 of Ch. XVI, *Analytic Geometry*, is all that is presupposed.

Rectangular arrays of this type, of which the square arrays are special cases, are called *matrices*. Thus

$$(2) \qquad \begin{Vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{Vmatrix}$$

is a matrix of two rows and three columns, and

$$(3) \qquad \begin{Vmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{Vmatrix}$$

is a *square* matrix with four rows and four columns.

A matrix, even when square, is not a determinant. The matrix (3) is the set of sixteen quantities arranged as shown, whereas the corresponding determinant is a single quantity, a polynomial formed from the sixteen quantities according to a certain law.

On the other hand, numerous determinants can be formed from a matrix by suppression of rows and columns. From the matrix (2) we have the three two-rowed determinants

$$\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}, \qquad \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}, \qquad \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix},$$

obtained by suppressing each of the columns in turn; and six one-rowed determinants, namely the elements themselves. Again, from the matrix (3) there can be formed determinants of the fourth, third, second, and first orders.

DEFINITION. *If, of the determinants which can be formed from a given matrix, not all those of order r are zero, whereas all those of order greater than r are zero, the matrix is said to be of rank r.*

For example, if, of the determinants that can be formed from a matrix of eight rows and six columns, all those of orders six, five, and four are zero, but at least one of order three is not zero, the rank of the matrix is three.

If the elements of a matrix are all zero, the definition cannot be applied. It is reasonable to agree that in this case the rank be zero.

The ranks of the matrices,

$$\begin{Vmatrix} 2 & 1 & 3 \\ 4 & 5 & 2 \end{Vmatrix}, \qquad \begin{Vmatrix} 4 & 2 & 6 \\ 6 & 3 & 9 \end{Vmatrix}, \qquad \begin{Vmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{Vmatrix},$$

are, respectively, 2, 1 and 0.

EXERCISES

Find the ranks of the following matrices.

1. $\begin{Vmatrix} 4 & -2 & 6 \\ -6 & 3 & -9 \end{Vmatrix}$. **2.** $\begin{Vmatrix} 1 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & 2 & 3 \end{Vmatrix}$. **3.** $\begin{Vmatrix} 1 & 2 & -1 & 3 \\ 4 & 3 & 2 & 1 \\ -2 & 1 & -4 & 5 \end{Vmatrix}$.

4. Show that the rank of the matrix obtained by interchanging the rows and columns in a given matrix is the same as that of the given matrix.

2. Homogeneous Linear Equations. A polynomial in x_1, x_2, x_3, all of whose terms are of the same degree in x_1, x_2, x_3, is said to be *homogeneous*. Thus

$$2x_1 - 5x_2 + 3x_3, \qquad 3x_1x_2 - 5x_2^2 + 4x_1x_3$$

are homogeneous polynomials in x_1, x_2, x_3 of the first and second degrees, respectively.

A. *System of Two Homogeneous Linear Equations in Three Unknowns.* Consider the simultaneous equations

$$\begin{aligned} a_1x_1 + a_2x_2 + a_3x_3 &= 0, \\ b_1x_1 + b_2x_2 + b_3x_3 &= 0. \end{aligned} \tag{1}$$

An obvious solution of these equations is $x_1 = 0$, $x_2 = 0$, $x_3 = 0$. This solution is never very useful in practice. In seeking other solutions, we distinguish three cases, according as the rank r of the matrix of the coefficients,

$$\begin{Vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{Vmatrix}, \tag{2}$$

is 2, 1, or 0.

Case 1: $r = 2$. At least one of the two-rowed determinants in (2) is not zero. Suppose that $a_1b_2 - a_2b_1 \neq 0$ and rewrite equations (1) in the form

$$\begin{aligned} a_1x_1 + a_2x_2 &= -a_3x_3, \\ b_1x_1 + b_2x_2 &= -b_3x_3. \end{aligned} \tag{3}$$

Considered as equations in x_1 and x_2 alone, these equations have, by Cramer's rule, the solution

$$x_1 = \frac{\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}}{\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}} x_3, \qquad x_2 = \frac{\begin{vmatrix} a_3 & a_1 \\ b_3 & b_1 \end{vmatrix}}{\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}} x_3, \tag{4}$$

no matter what the value of x_3 may be. Hence equations (1), in

x_1, x_2, x_3, have the solution

$$(5) \qquad x_1 = \frac{|a_2\, b_3|}{|a_1\, b_2|}\, x_3, \qquad x_2 = \frac{|a_3\, b_1|}{|a_1\, b_2|}\, x_3, \qquad x_3 = x_3,$$

where x_3 is an arbitrary constant.*

If k is an arbitrary constant, $k|a_1\, b_2|$ is also an arbitrary constant. Hence we may replace x_3 in (5) by $k|a_1\, b_2|$. The result is

$$(\mathrm{I}) \qquad x_1 = k|a_2\, b_3|, \qquad x_2 = k|a_3\, b_1|, \qquad x_3 = k|a_1\, b_2|,$$

where k is an arbitrary constant.

Inasmuch as all the solutions of (3) are given by (4), all the solutions of (1) are given by (I).

Since (I) comprises all the solutions of (1), it is known as *the general solution* of (1). A solution obtained from it by giving to k a specific value is called *a particular solution.* The particular solution for $k = 0$ is 0, 0, 0, and that for $k = 1$ is $|a_2\, b_3|$, $|a_3\, b_1|$, $|a_1\, b_2|$.

Case 2: $r = 1$. All the two-rowed determinants in (2) vanish, but at least one element is not zero. Assume, for example, that $a_1 \neq 0$ and solve the first equation for x_1:

$$x_1 = -\frac{a_2}{a_1}x_2 - \frac{a_3}{a_1}x_3.$$

This value of x_1 satisfies the second equation, considered as an equation in x_1, no matter what the values of x_2 and x_3 are, for the result of substituting it in the second equation is

$$(a_1b_2 - a_2b_1)x_2 + (a_1b_3 - a_3b_1)x_3 = 0,$$

and $a_1b_2 - a_2b_1$ and $a_1b_3 - a_3b_1$ are both zero, by hypothesis.

It follows that the general solution of equations (1) is given by

$$(6) \qquad x_1 = -\frac{a_2}{a_1}x_2 - \frac{a_3}{a_1}x_3, \qquad x_2 = x_2, \qquad x_3 = x_3,$$

or by

$$(\mathrm{II}) \qquad x_1 = -k\,a_2 - l\,a_3, \qquad x_2 = k\,a_1, \qquad x_3 = l\,a_1,$$

where k and l are arbitrary constants.

Case 3: $r = 0$. Since in this case all the coefficients in (1) are zero, the general solution is

$$(\mathrm{III}) \qquad x_1 = k, \qquad x_2 = l, \qquad x_3 = m,$$

where k, l, and m are arbitrary constants.

* Each of the determinants in (4) we have now represented by the elements of its principal diagonal, enclosed by bars.

In all three cases the two equations have solutions other than 0, 0, 0. This is true also of a single equation, as may be readily verified.

THEOREM A. *A system of less than three homogeneous linear equations in three unknowns always has solutions other than* 0, 0, 0.

Inspection of the three cases shows that *the general solution of the system* (1) *always contains* $3 - r$ *arbitrary constants.* The values of $3 - r$ unknowns may be arbitrarily assigned and those of the remaining unknowns are then determined. Thus, the general solution (II) in the case $r = 1$ contains the two arbitrary constants, k and l, corresponding to arbitrary values assigned to x_2 and x_3.

EXERCISE. Show that the italicized statement is true also of a single equation $a_1x_1 + a_2x_2 + a_3x_3 = 0$.

B. *System of Three Homogeneous Linear Equations in Three Unknowns.* The equations,

$$\begin{aligned} a_1x_1 + a_2x_2 + a_3x_3 &= 0, \\ b_1x_1 + b_2x_2 + b_3x_3 &= 0, \\ c_1x_1 + c_2x_2 + c_3x_3 &= 0, \end{aligned} \tag{7}$$

do not always have solutions other than 0, 0, 0. For, if the determinant $|a\,b\,c|$ of the coefficients is not zero, the system has, according to § 1, just one solution, given by Cramer's rule, and this solution, since the right-hand members in (7) are all zero, is 0, 0, 0.

It follows that, if equations (7) are to have solutions different from 0, 0, 0, the determinant of the coefficients must vanish. Conversely, if $|a\,b\,c| = 0$, the equations actually have solutions other than 0, 0, 0, for the rank r of the matrix $\|a\,b\,c\|$ is then 2, 1, or 0, and we shall show that in these three cases equations (7) behave precisely as did equations (1) in the three corresponding cases.

Case 1: $r = 2$. We may assume, for example, that $|a_1\,b_2| \neq 0$. The general solution of the first two equations in (7) is, then, by Case 1 under A,

$$x_1 = k|a_2\,b_3|, \qquad x_2 = k|a_3\,b_1|, \qquad x_3 = k|a_1\,b_2|. \tag{I}$$

The result of substituting these values of x_1, x_2, x_3 in the third equation is

$$k|a\,b\,c| = 0,$$

and $|a\,b\,c| = 0$, by hypothesis. Hence all the solutions of (7) are given by (I).

Case 2: $r = 1$. In this case, at least one coefficient in (7) is not zero, say $a_1 \neq 0$. The general solution of (7) is then (II), for the values in (II) constitute the general solution of the first equation, and they satisfy the remaining equations by virtue of the fact that all the two-rowed determinants formed from $\| a\ b\ c \|$ are zero; see Case 2 under A.

Case 3: $r = 0$. Here, all the coefficients in (7) are zero and the general solution is given by (III).

The proof of our contention is now complete.

THEOREM B. *A necessary and sufficient condition that three homogeneous linear equations in three unknowns have a solution other than* 0, 0, 0 *is that the determinant of the coefficients of the unknowns vanish.*

Here, too, it is true that, if the rank of the system, that is, the rank of the matrix $\| a\ b\ c \|$, is r, *the general solution contains* $3 - r$ *arbitrary constants.* For, if $r < 3$, the solutions of (7) are in each case precisely those of (1); and, if $r = 3$, the general solution of (7) is the single solution 0, 0, 0 and contains no arbitrary constant.

C. *System of More than Three Homogeneous Linear Equations in Three Unknowns.* Imagine that to equations (7) there are adjoined additional equations of the same type. The resulting system has a matrix whose rank r is at most three.

If $r = 3$, at least one of the three-rowed determinants in the matrix is not zero. Suppose that $|a\ b\ c| \neq 0$. Then the first three equations have as their only solution 0, 0, 0, and hence this is the only solution of the system.

If $r = 2$, and we assume, in particular, that $|a_1\ b_2| \neq 0$, the general solution of the system is (I), for the values in (I) constitute the general solution of the first two equations, and they satisfy the remaining equations by virtue of the fact that all the three-rowed determinants in the original matrix are zero; see Case I under B.

If $r = 1$ and $a_1 \neq 0$, (II) is the general solution of the system; and, if $r = 0$, the general solution is given by (III).

From this, and the previous, discussion we conclude the theorem:

THEOREM C. *A system of three or more homogeneous linear equations in three unknowns has solutions other than* 0, 0, 0 *if and only if the rank of the matrix of the coefficients is less than three.*

We note that in this case, as well as in the preceding cases, the general solution always contains $3 - r$ arbitrary constants.

Summary. We may summarize our theoretical results in one simple theorem.

THEOREM 1. *If the rank of a system of homogeneous linear equations in three unknowns is r, the general solution of the system contains $3 - r$ arbitrary constants.*

The theorem says that, when $r = 3$, the general solution contains no arbitrary constant. This means that the only solution is 0, 0, 0; for, if there were a solution x_1, x_2, x_3 other than 0, 0, 0, then $k\,x_1, k\,x_2, k\,x_3$, where k is an arbitrary constant, would be a solution.

Since the theorem guarantees 0, 0, 0 as the only solution when $r = 3$, and implies solutions other than 0, 0, 0 when $r < 3$, it actually covers the three previous theorems. Thus, if there are fewer than three equations, r is always less than three and there are always solutions other than 0, 0, 0 (Theorem A). If there are at least three equations, r may be equal to three or less than three, and only in the latter case are there solutions other than 0, 0, 0 (Theorems B, C).

General Case. The results obtained for systems of equations in three unknowns are typical. They admit immediate extension to the general case of a system in any given number of unknowns.

THEOREM 2. *If the rank of a system of homogeneous linear equations in n unknowns is r, the general solution of the system contains $n - r$ arbitrary constants.*

COROLLARY A. *If the number of equations is less than n, the system always has solutions other than $0, 0, \cdots, 0$.*

COROLLARY B. *If the number of equations is equal to n, the system has solutions other than $0, 0, \cdots, 0$ if and only if the determinant of the coefficients vanishes.*

COROLLARY C. *If the number of equations is equal to or greater than n, the system has solutions other than $0, 0, \cdots, 0$ if and only if $r < n$.*

EXERCISES

1. Solve the system of equations

$$2x_1 + 3x_2 + 6x_3 = 0, \qquad 3x_1 - 6x_2 + 2x_3 = 0.$$

Find a particular solution and then write the general solution.

2. Find all the solutions of the system

$$3x_1 + 2x_2 - 2x_3 = 0, \qquad 2x_1 + 3x_2 - x_3 = 0, \qquad 8x_1 + 7x_2 - 5x_3 = 0.$$

3. Prove Cor. B of Th. 2 when $n = 2$.

4. Prove Cor. C of Th. 2 for the system of equations

$$a_1x_1 + a_2x_2 = 0, \qquad b_1x_1 + b_2x_2 = 0, \qquad c_1x_1 + c_2x_2 = 0.$$

5. Discuss completely a system of three linear homogeneous equations in four unknowns. Then prove Cor. B of Th. 2 for $n = 4$.

6. Find all the solutions of the system

$$x_1 + 2x_2 - x_3 + x_4 = 0, \qquad x_1 + 2x_2 + x_3 - 2x_4 = 0.$$

7. Show that a system of homogeneous linear equations in n unknowns has a solution other than $0, 0, \cdots, 0$ if and only if its rank is less than n.

3. Proportionality. Linear Dependence. The ordinary definition of proportionality of two pairs of numbers demands that *each* number pair be a multiple of the other. The number pairs 2, 4 and 3, 6 are proportional according to this definition. On the other hand, the number pairs 2, 4 and 0, 0 are not proportional according to it; for, though the second is a multiple of the first, the first is not a multiple of the second. It is convenient to extend the definition so that pairs of numbers of this type will be proportional, by demanding merely that *at least one* of the number pairs be a multiple of the other.

We formulate the extended definition as follows.

DEFINITION 1. *The number pairs a_1, a_2 and b_1, b_2 are proportional if there exist two numbers, k and l, not both zero, such that*

$$(1) \qquad k\,a_1 + l\,b_1 = 0, \qquad k\,a_2 + l\,b_2 = 0.$$

Since it is required that at least one of the numbers k, l be not zero, the definition does demand that at least one of the number pairs be a multiple of the other.*

An important advantage of the extended definition is easily cited. If a_1, a_2 and b_1, b_2 are proportional in either sense, their determinant is zero: $|a\,b| = 0$. The converse is not true, if we hold to the original definition, as the case of the number pairs 2, 4 and 0, 0 shows. It is true, however, when the extended definition is adopted.

THEOREM 1. *The number pairs a_1, a_2 and b_1, b_2 are proportional (in the extended sense) if and only if their determinant vanishes: $|a\,b| = 0$.*

Definition 1 says that a_1, a_2 and b_1, b_2 are proportional if and only if equations (1) in k, l have a solution other than 0, 0. But two homogeneous linear equations in two unknowns have a solution other

* For example, if $k \neq 0$, (1) can be rewritten $a_1 = m\,b_1$, $a_2 = m\,b_2$, where $m = -l/k$.

than 0, 0 when and only when the determinant of their coefficients vanishes. Hence the theorem is proved.

The definition of proportionality can be readily extended to the case of two sets of n numbers each. For example, when $n = 3$, we have:

DEFINITION 2. *The number triples*

$$\begin{matrix} a_1, & a_2, & a_3, \\ b_1, & b_2, & b_3, \end{matrix} \tag{2}$$

are proportional if and only if two numbers, k and l, not both zero, exist so that

$$k\,a_1 + l\,b_1 = 0, \qquad k\,a_2 + l\,b_2 = 0, \qquad k\,a_3 + l\,b_3 = 0. \tag{3}$$

In other words, the number triples are proportional when and only when the system (3) of three homogeneous linear equations in the two unknowns k, l have a solution other than 0, 0. But a condition, necessary and sufficient that such a solution exist, is that the rank of the system be less than two; see § 2, Th. 2, Cor. C.

THEOREM 2. *The two number triples a_1, a_2, a_3 and b_1, b_2, b_3 are proportional if and only if*

$$\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} = 0, \qquad \begin{vmatrix} a_3 & a_1 \\ b_3 & b_1 \end{vmatrix} = 0, \qquad \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = 0, \tag{4}$$

that is, if and only if the rank of their matrix is less than two.

Linear Dependence. Proportionality of two sets of numbers is a special type of dependence of the two sets upon one another. Equations exhibiting this dependence are linear. Herein lies the motivation of the term "linear dependence," which, when applied to *two* sets of numbers, is synonymous with "proportionality."

We proceed to extend the idea of linear dependence in applying it to three number triples.

DEFINITION 3. *The number triples*

$$\begin{matrix} a_1, & a_2, & a_3, \\ b_1, & b_2, & b_3, \\ c_1, & c_2, & c_3, \end{matrix} \tag{5}$$

are linearly dependent if and only if three numbers k, l, m, not all zero, exist so that

$$\begin{aligned} k\,a_1 + l\,b_1 + m\,c_1 &= 0, \\ k\,a_2 + l\,b_2 + m\,c_2 &= 0, \\ k\,a_3 + l\,b_3 + m\,c_3 &= 0. \end{aligned} \tag{6}$$

Equations (6) have a solution for k, l, m, other than 0, 0, 0, if and only if $|a\,b\,c| = 0$. Hence:

THEOREM 3. *Three number triples are linearly dependent if and only if their determinant vanishes; or, if and only if the rank of their matrix is less than three.*

The numbers k, l, m in this case, or k, l in the previous cases, are known as the *constants of dependence.* They may frequently be found by inspection, and can always be found by solving the equations which express the linear dependence.

We have seen that two number triples are linearly dependent if and only if the rank of their matrix is less than two, and that three are linearly dependent if and only if the rank of their matrix is less than three. In general:

THEOREM 4. *A necessary and sufficient condition that m number triples be linearly dependent is that the rank of their matrix be less than m.*

Since the matrix of the triples always has three columns, its rank can never be greater than three and consequently is always less than m, when $m > 3$. Hence the theorem implies that *more than three number triples are always linearly dependent.*

Let us prove that this is the case when $m = 4$. The three number triples (5) together with a fourth, d_1, d_2, d_3, are, by definition, linearly dependent if and only if four numbers A, B, C, D, not all zero, exist so that

$$
\begin{aligned}
Aa_1 + Bb_1 + Cc_1 + Dd_1 &= 0,\\
Aa_2 + Bb_2 + Cc_2 + Dd_2 &= 0,\\
Aa_3 + Bb_3 + Cc_3 + Dd_3 &= 0.
\end{aligned}
\tag{7}
$$

These three equations in the four unknowns A, B, C, D always have a solution other than 0, 0, 0, 0; see § 2, Th. 2, Cor. A. Hence four number triples are always linearly dependent.

By similar reasoning we could show that five or more triples are always linearly dependent. We prefer, however, to prove this in another way. Let the triples be, for example, the previous four and e_1, e_2, e_3. The previous four we know are linearly dependent: A, B, C, D, not all zero, exist so that equations (7) are valid. Then A, B, C, D, E, not all zero, exist so that the three equations

$$Aa_i + Bb_i + Cc_i + Dd_i + Ee_i = 0, \qquad (i = 1, 2, 3),$$

are valid; we have but to retain the previous values for A, B, C, D, and take E as 0.

General Case. The generalization of our results to the case of sets of n numbers each is simple.

THEOREM 5. *A necessary and sufficient condition that m sets of n numbers each be linearly dependent is that the rank of their matrix be less than m. In particular, if $m > n$, the sets are always linearly dependent. If $m = n$, they are linearly dependent if and only if their determinant vanishes.*

EXERCISES

1. Show that the number triples 5, 14, 4; 2, -1, 1; 3, 4, 2 are linearly dependent and find values for the constants of dependence.

2. Formulate the definition of linear dependence for four sets of four numbers each. Show that a condition, necessary and sufficient that they be linearly dependent, is that their determinant vanish.

3. Are the sets of numbers 1, 2, 4, 3; 2, 2, 1, 3; 4, 1, 2, 5 linearly dependent?

4. Formulate the definition of linear dependence for m sets of n numbers each. Hence establish Theorem 5.

5. Show that if there exists, among m sets of n numbers each, a smaller number of sets which are linearly dependent, the m sets are linearly dependent.

6. Prove that, if any one of m sets of n numbers each consists exclusively of zeros, the m sets are linearly dependent.

4. Linear Combination. The number triple c_1, c_2, c_3 is called a linear combination of the number triples a_1, a_2, a_3 and b_1, b_2, b_3 if two numbers k, l exist so that

$$(1) \qquad c_1 = k\,a_1 + l\,b_1, \qquad c_2 = k\,a_2 + l\,b_2, \qquad c_3 = k\,a_3 + l\,b_3.$$

In general, the number set $h_1, h_2, \cdots, h_n$ is a linear combination of the number sets $a_1, a_2, \cdots, a_n$; $b_1, b_2, \cdots, b_n$; $\cdots\cdots$; $g_1, g_2, \cdots, g_n$ if numbers $A, B, \cdots, G$ can be found so that the n equations

$$(2) \qquad h_i = Aa_i + Bb_i + \cdots + Gg_i, \qquad (i = 1, 2, \cdots, n),$$

subsist.

Linear combination and linear dependence are closely related. For example, equations (1) can be rewritten in a form,

$$k\,a_1 + l\,b_1 - c_1 = 0, \qquad k\,a_2 + l\,b_2 - c_2 = 0, \qquad k\,a_3 + l\,b_3 - c_3 = 0,$$

which tells us that the triples of a's, b's, and c's are linearly dependent. Again, equations (2) may be interpreted as expressing the linear dependence of $a_1, a_2, \cdots, a_n$; $b_1, b_2, \cdots, b_n$; $\cdots\cdots$; $g_1, g_2, \cdots, g_n$; and $h_1, h_2, \cdots, h_n$.

THEOREM 1. *If a number set is a linear combination of m number sets, the $m + 1$ number sets are linearly dependent.*

The converse of the theorem is true.

THEOREM 2. *If certain number sets are linearly dependent, at least one set among them is a linear combination of the others.*

Consider, say, equations (7) of § 3, expressing the linear dependence of the number triples a_1, a_2, a_3; b_1, b_2, b_3; c_1, c_2, c_3; d_1, d_2, d_3. Of the constants of dependence A, B, C, D, at least one is not zero. Suppose that $D \neq 0$. Equations (7) may, then, be solved for d_1, d_2, d_3, and the triple of d's is thus obtained as a linear combination of the other three triples.

EXERCISES

1. Show that

$$\begin{vmatrix} a_1 & 3a_1 - c_1 & c_1 \\ a_2 & 3a_2 - c_2 & c_2 \\ a_3 & 3a_3 - c_3 & c_3 \end{vmatrix} = 0.$$

2. Prove that, if one column, or row, of a determinant is a linear combination of two or more columns, or rows, the value of the determinant is zero.

3. Prove that, if the sum of the elements in each column, or row, of a determinant is zero, the determinant is zero.

5. Homogeneous Linear Equations. Conclusion.

THEOREM 1. *If $r_1, r_2, \cdots, r_n$ and $s_1, s_2, \cdots, s_n$ are two solutions of a system of homogeneous linear equations in n unknowns, $x_1, x_2, \cdots, x_n$, then*

$$(a) \qquad k r_1, \quad k r_2, \quad \cdots, \quad k r_n,$$

$$(b) \qquad r_1 + s_1, \quad r_2 + s_2, \quad \cdots, \quad r_n + s_n,$$

where k is arbitrary, are also solutions.

The proof of the theorem is left to the reader. From it follows that, if $r_1, r_2, \cdots, r_n$ and $s_1, s_2, \cdots, s_n$ are solutions, $k r_1, k r_2, \cdots, k r_n$ and $l s_1, l s_2, \cdots, l s_n$ are solutions, and therefore

$$(c) \qquad k r_1 + l s_1, \quad k r_2 + l s_2, \quad \cdots, \quad k r_n + l s_n$$

is a solution. Similarly, if $t_1, t_2, \cdots, t_n$ is a third solution,

$$(d) \quad k r_1 + l s_1 + m t_1, \quad k r_2 + l s_2 + m t_2, \quad \cdots, \quad k r_n + l s_n + m t_n$$

is a solution.

Solution (c) is a linear combination of two particular solutions, and solution (d) is a linear combination of three particular solutions. In general:

A linear combination of a number of solutions is a solution.

We illustrate these developments by reinterpreting the results obtained in § 2 for the system

$$a_1x_1 + a_2x_2 + a_3x_3 = 0,$$
$$b_1x_1 + b_2x_2 + b_3x_3 = 0.$$

The general solution of the system in case $r = 0$, namely,

$$x_1 = k, \qquad x_2 = l, \qquad x_3 = m,$$

may be written in the form

$$\begin{aligned} x_1 &= k\,(1) + l\,(0) + m\,(0), \\ x_2 &= k\,(0) + l\,(1) + m\,(0), \\ x_3 &= k\,(0) + l\,(0) + m\,(1), \end{aligned}$$

and hence is an arbitrary linear combination of the three particular solutions $1, 0, 0$; $0, 1, 0$; $0, 0, 1$. It is to be noted that these particular solutions are not linearly dependent.

In case $r = 1$, the general solution

$$x_1 = -a_2k - a_3l, \qquad x_2 = ka_1, \qquad x_3 = la_1, \qquad a_1 \neq 0,$$

or

$$x_1 = k\,(-a_2) + l\,(-a_3), \qquad x_2 = k\,(a_1) + l\,(0), \qquad x_3 = k\,(0) + l\,(a_1)$$

is an arbitrary linear combination of the two particular solutions $-a_2, a_1, 0$ and $-a_3, 0, a_1$. Here, too, the particular solutions are linearly independent.

Finally, if $r = 2$, the general solution

$$x_1 = k|a_2\,b_3|, \qquad x_2 = k|a_3\,b_1|, \qquad x_3 = k|a_1\,b_2|$$

is a linear combination (multiple) of the particular solution $|a_2\,b_3|$, $|a_3\,b_1|$, $|a_1\,b_2|$.

These results suggest the following proposition.

THEOREM 2. *If the rank of a system of homogeneous linear equations in n unknowns is r, every solution of the system can be written as a linear combination of $n - r$ linearly independent solutions.*

In each example cited, the set of solutions from which the general solution was formed by the process of linear combination was a special set of linearly independent solutions. As a matter of fact, Theorem 2 remains true when any set of $n - r$ linearly independent solutions is employed.

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1. Show that

$$\begin{vmatrix} a_1 & 3a_1 - c_1 & c_1 \\ a_2 & 3a_2 - c_2 & c_2 \\ a_3 & 3a_3 - c_3 & c_3 \end{vmatrix} = 0.$$

2. Prove that, if one column, or row, of a determinant is a linear combination of two or more columns, or rows, the value of the determinant is zero.

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$$(b) \qquad r_1 + s_1, \quad r_2 + s_2, \quad \cdots, \quad r_n + s_n,$$

where k is arbitrary, are also solutions.

The proof of the theorem is left to the reader. From it follows that, if $r_1, r_2, \cdots, r_n$ and $s_1, s_2, \cdots, s_n$ are solutions, $k\,r_1, k\,r_2, \cdots, k\,r_n$ and $l\,s_1, l\,s_2, \cdots, l\,s_n$ are solutions, and therefore

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may be written in the form

$$\begin{aligned} x_1 &= k\,(1) + l\,(0) + m\,(0), \\ x_2 &= k\,(0) + l\,(1) + m\,(0), \\ x_3 &= k\,(0) + l\,(0) + m\,(1), \end{aligned}$$

and hence is an arbitrary linear combination of the three particular solutions 1, 0, 0; 0, 1, 0; 0, 0, 1. It is to be noted that these particular solutions are not linearly dependent.

In case $r = 1$, the general solution

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or

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is an arbitrary linear combination of the two particular solutions $-a_2$, a_1, 0 and $-a_3$, 0, a_1. Here, too, the particular solutions are linearly independent.

Finally, if $r = 2$, the general solution

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is a linear combination (multiple) of the particular solution $|a_2\,b_3|$, $|a_3\,b_1|$, $|a_1\,b_2|$.

These results suggest the following proposition.

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In each example cited, the set of solutions from which the general solution was formed by the process of linear combination was a special set of linearly independent solutions. As a matter of fact, Theorem 2 remains true when any set of $n - r$ linearly independent solutions is employed.

EXERCISES

1. Prove Theorem 1 for $n = 3$.

2. The rank of the matrix of the system of equations

$$x_1 - x_2 + x_3 + x_4 = 0,\quad x_1 + x_2 + 5x_3 - 3x_4 = 0,\quad 3x_1 - x_2 + 7x_3 - x_4 = 0$$

is two. Find the general solution of the system, and show that it can be considered as an arbitrary linear combination of two particular, linearly independent solutions.

6. Determinants and Their Cofactors. The reader will recall that the minor $\mathfrak{M}$ of an element m in a determinant Δ is the determinant obtained from Δ by striking out the row and column in which m stands. He will also remember that, if m is in the ith row and jth column, the product

$$(1)\qquad (-1)^{i+j}\, m\mathfrak{M}$$

consists of all the terms of Δ which contain m, and that the sum of the products of this type formed for all the elements of a row, or column, is equal to Δ.

We shall find it convenient to attach the sign prefixed to $m\mathfrak{M}$ in (1) directly to the minor $\mathfrak{M}$ itself and to give to the signed minor, $(-1)^{i+j}\mathfrak{M}$, a new name: *cofactor.*

Definition. *The cofactor, M, of the element m is the signed minor of m:*

$$M = (-1)^{i+j}\mathfrak{M}.$$

It is now the product mM of an element m by its cofactor M which consists of all the terms of Δ which contain the element. Hence:

Theorem 1. *The sum of the products of the elements of a row, or column, of a determinant by their cofactors is equal to the determinant.*

Theorem 2. *If each element of a row, or column, of a determinant is multiplied by the cofactor of the corresponding element of a different row, or column, the sum of the resulting products is zero.*

For a more intensive treatment of the subjects discussed in this chapter, the reader is referred to Bôcher, *Introduction to Higher Algebra,* Chs. II, III, IV.

EXERCISES

1. The determinant obtained from a given determinant by replacing each element by its cofactor is called the *adjoint* of the given determinant. If Δ is a determinant of the nth order and Δ' is its adjoint, show that

$$(a)\ \text{if } n = 2,\quad \Delta' = \Delta; \qquad (b)\ \text{if } n = 3,\quad \Delta' = \Delta^2.$$

What is the value of Δ' in the general case?

2. *If M is the cofactor of an element m in a determinant Δ of the nth order and μ is the cofactor of M in the adjoint of Δ, then*

Verify this theorem (*a*) when $n = 2$; (*b*) when $n = 3$.

3. A determinant is said to be *symmetric* if each two elements symmetrically situated with respect to the principal diagonal are equal.

If a determinant is symmetric, its adjoint is symmetric.

Verify this theorem when $n = 3$.

4. A determinant is said to be *skew-symmetric* if each element in the principal diagonal is zero, and each two elements symmetrically situated with respect to the principal diagonal are negatives of one another.

A skew-symmetric determinant of odd order is always zero.

Verify this theorem when $n = 3$.

5. *The product of two determinants of the nth order may be expressed as a determinant of the nth order in which the element in the ith row and jth column is the sum of the products of the corresponding elements in the ith row of the first and the jth column of the second determinant.*

Verify this theorem in the case $n = 2$:

$$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \cdot \begin{vmatrix} \alpha_1 & \alpha_2 \\ \beta_1 & \beta_2 \end{vmatrix} = \begin{vmatrix} a_1\alpha_1 + a_2\beta_1 & a_1\alpha_2 + a_2\beta_2 \\ b_1\alpha_1 + b_2\beta_1 & b_1\alpha_2 + b_2\beta_2 \end{vmatrix}.$$

CHAPTER II

GEOMETRICAL INTRODUCTION

1. Projections and Rigid Motions. Projective and Metric Properties. The geometry of Euclid, with which the reader is familiar, concerns itself with lengths, angles, and areas; it is a geometry of measure, or a *metric* geometry. In contrast to it, there was developed about a hundred years ago a new kind of geometry, known as *projective* geometry, which has nothing to do with measurement.

In order to understand the fundamental difference between the two kinds of geometry, we shall study, first, two corresponding kinds of operations, known as rigid motions and projections.

Projection of a Line upon a Line. Let L and L' be two distinct lines in the same plane, and let O be a point of the plane not lying on either line. Let $A, B, C, D, \cdots$ be points of L, and let $A', B', C', D', \cdots$ be the points of L' in which the lines $OA, OB, OC, OD, \cdots$ meet L'. Then $A', B', C', D', \cdots$ are called the projections on L' of the points $A, B, C, D, \cdots$ of L from the point O, and the process described is known as the projection from O of the line L on the line L'. The point O is called the center of projection.

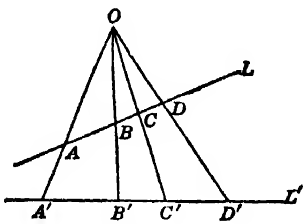


FIG. 1

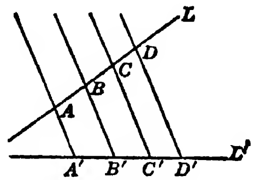


FIG. 2

Instead of using the lines through a point to effect the projection of L on L' (Fig. 1), we can use the lines with a given direction (Fig. 2), provided merely that the direction chosen is distinct from those of L and L'. In this case, the projection is called a *parallel* projection; in the previous case, a *central* projection.

Projection of a Plane upon a Plane. Let p and p' be two distinct planes in space, and let O be a point in space not lying in either plane.

In the *central* projection from O of the plane p on the plane p', an arbitrary point P of p is projected into the point P' of p' in which the line OP meets the plane p' (Fig. 3).

Again, let d be a direction in space not parallel to either of the planes p or p'. In the *parallel* projection, in the direction d, of p on p', an arbitrary point P of p is projected into the point P' of p' in which the line through P with direction d meets p'.

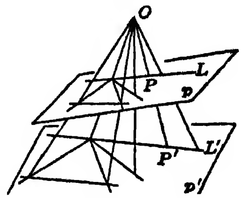


FIG. 3

A straight line L in p determines with O (Fig. 3) a plane which intersects p' in a straight line L'. If a point P traces L, its projection P' traces L'. Hence a straight line projects by a central projection—and also by a parallel projection, as is readily seen—into a straight line.

Evidently, points which are collinear, that is, lie on a line, project into points which are collinear; lines which are concurrent, that is, go through a point, project into lines which are concurrent; and a triangle projects into a triangle.

The student is familiar with the fact that the plane sections of a cone of revolution are conics. This is also true of any cone obtained by joining the points of a conic with a point not in the plane of the conic, or of any cylinder obtained by drawing through the points of a conic parallel lines not lying in the plane of the conic. It follows, then, that a conic projects into a conic.

We have enumerated various properties of figures which are unchanged by projections. These properties are known as *projective* properties.

DEFINITION. *A property of a figure which is preserved by* EVERY *projection is called a projective property.*

The following properties we have found to be projective: that a curve be a straight line; that a curve be a conic; that a point lie on a straight line; that a number of points be collinear; that a number of lines be concurrent; that a rectilinear figure be a triangle.

It is important to note that, in order that a property be projective, it must be preserved, not only by some projections, but by *all* projections. For example, it is possible to find projections which carry a circle into a circle; but not every projection will do so. Hence the property that a curve be a circle is not a projective property.

Rigid Motions. Whenever we move an object from one place to another without in any way altering the object itself, we have subjected the object to a rigid motion. Thus, a rigid motion effects a change of the position of the object in space.

The properties which we have enumerated as being projective are obviously preserved by rigid motions. In fact, every projective property is preserved by rigid motions. There are, however, many nonprojective properties which are unchanged by rigid motions, for example, distance, angle, and area. These properties are called *metric* properties.

DEFINITION. *A property which is preserved by all rigid motions, but not by every projection, is a metric property.*

A circle is a metric figure, whereas the general conic is projective. Again, an isosceles triangle is metric; an arbitrary triangle, projective.

Projective and Metric Theorems. Just as we have characterized properties as projective and metric, so also can we differentiate theorems.

DEFINITION. *A theorem which deals merely with projective properties is a projective theorem. A theorem into which metric properties enter, either alone or in conjunction with projective properties, is a metric theorem.*

The Pythagorean theorem is a good example of a metric theorem. In fact, all the theorems of Euclidean geometry are metric. Euclidean geometry is a geometry of metric properties, or a *metric geometry.*

Projective geometry deals merely with projective properties. It makes a study of these properties and seeks to establish relationships between them, in the form of theorems, in the same general way in which Euclidean geometry treats metric properties and sets up theorems concerning them.*

Geometries. The idea of classifying properties of figures according to the manner in which they behave when the figures are subjected to certain operations was conceived about fifty years ago by the German mathematician, Felix Klein (1849–1925). Our present discussion can

* Though isolated projective theorems and theories existed before his time, the real founder of projective geometry was the French soldier, statesman, and mathematician, Poncelet (1789–1867). His principal work on the subject, *Traité des Propriétés Projectives des Figures*, published in 1822, was developed during the years (1812–14) which he spent as a prisoner of war in a Russian prison.

be considered only a brief introduction to this idea. We shall seek later to develop it more precisely, and to extend its range of application. One thing, however, it has already taught us: There is not just one geometry. In fact, besides Euclidean geometry and projective geometry, there are, as we shall see later, numerous other geometries.

EXERCISES

1. Which of the following figures are projective? Which are metric?

(*a*) A parallelogram; (*b*) an acute-angled triangle; (*c*) a figure consisting of four points, no three on a line, and the six lines joining them; (*d*) a triangle and a median line; (*e*) a parabola; (*f*) a conic and a number of points marked on it; (*g*) the same, when one of the points is a vertex of the conic; (*h*) a line and a conic with two points in common.

2. Show that the property that a point be midway between two points is preserved by every parallel projection and by every central projection in which the two lines L and L' are parallel. Is the property metric or projective?

3. Prove that the property that a line be tangent to a curve is a projective property.

2. Vanishing Points and Lines of Projections. In the central projection shown in Fig. 4, a point on L will fail to have a projection on L' when the line joining it to O is parallel to L'. There is just one point V on L with this property. Similarly, there is one point on L' which is not the projection of a point on L, the point W' in which L' is met by the parallel to L through O.

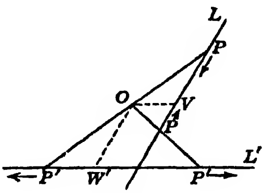


Fig. 4

When a point P on L approaches the point V as a limit, first from the one side and then from the other, its projection P' recedes on L' indefinitely, first in the one direction and then in the other. For this reason, V is known as the *vanishing point* on L. Similarly, W' is the vanishing point on L'.

A point on L, other than V, projects into a definite point on L'; conversely, each point on L', other than W', is the projection of a definite point on L. Hence, if we exclude V and W', to each point on L corresponds one point on L', and to each point on L' corresponds one point on L. We express these facts by saying that the projection establishes a *one-to-one correspondence* between the points of L, other than V, and the points of L', other than W'.

In the central projection of Fig. 5, a line in p is a *vanishing line*, that is, fails to have a projection on p', when the plane determined by it and O is parallel to p'. There is a unique plane through O parallel to p' and hence a unique vanishing line in p, the line V in which this plane meets p. Likewise, there is a single vanishing line, W', in p'.

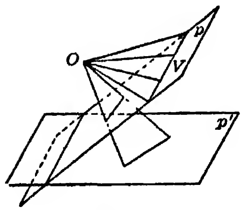


FIG. 5

The points of the vanishing lines are vanishing points, and the only vanishing points. For a point, say, in p is a vanishing point if and only if the line joining it to O is parallel to p'. But all the lines through O parallel to p' lie in the plane through O parallel to p', and the points in which they intersect p are precisely the points of V.

The projection establishes a one-to-one correspondence between the points of the plane p, other than those on V, and the points of the plane p', other than those on W'. It also establishes a one-to-one correspondence between the lines of p, other than V, and the lines of p', other than W'.

In view of the foregoing discussion, we must qualify some of the statements made in the previous paragraph. We said there, for example, that a straight line projects into a straight line. This is true, in general. A vanishing line is, however, an exception, for it has no projection. Again, two intersecting lines neither of which is the vanishing line, project into two intersecting lines, not always, but only in general. They may, by exception, project into two parallel lines; see Ex. 3.

EXERCISES

1. Describe all the projections of one line upon another in which there are no vanishing points.

2. Describe all the projections of one plane upon another in which there are no vanishing points.

3. Prove that two lines in the plane p (Fig. 5) which intersect on the vanishing line project into two parallel lines.

4. A triangle projects in general into a triangle. What are the exceptions?

3. Extensions of the Plane and of Space. Higher Geometry sets itself the task of devising means to dissolve exceptional cases such as those encountered in the preceding paragraph. It seeks, not to set the

exceptional cases aside—that is impossible—but to reshape its ideas so that the exceptional cases will actually be embraced under the general cases, as particular instances of them.

The exceptions of the previous paragraph can be traced to a single cause: the failure of two parallel lines to intersect. If we can remove this cause, the exceptions will be removed with it.

If two parallel lines are to have a point in common, this point must be a new point, other than those with which we are familiar. We are certainly at liberty to create new points, if it pleases and benefits us. We agree, then, to conceive a new point common to the two parallel lines, and to call it their point of intersection. This new point is an *ideal* point in that it has no *real* existence in the sense that the points with which we are familiar may be said to have. It is customary, for obvious reasons, to call it *a point at infinity.*

The rails of a straight railroad track appear to converge in both directions. Shall we then conceive two points at infinity on our parallel lines, one for each direction?

The question is not whether we can or cannot, but whether it would serve our purpose. It obviously would not, for then we should have the two lines intersecting, not in one point, but in two points, and two intersections are as unwelcome as none.

In the central projection of Fig. 4, we should naturally make correspond to the point V on L the point or points at infinity on L', and if we conceived two points at infinity on L', to V would correspond two points. Here, again, two are as bad as none, for the projection orders to each point on L, other than V, just one point of L'.

On each line it is convenient to create just one ideal point.

It is natural to think of the points at infinity on all the lines with a given direction as the same point, and to call this point *the point at infinity in the given direction.*

Let us now summarize the results of our discussion.

AGREEMENT 1. *Corresponding to each set of lines consisting of all the lines with a given direction, there is created an ideal point, or, point at infinity. This point shall be thought of as lying on each line of the set and on no other line. It shall be known as the point at infinity in the given direction.*

Inasmuch as there are infinitely many directions in the plane, there are infinitely many ideal points. It is natural to think of these points

as constituting a curve. But what type of curve? A circle? An ellipse? Or what?

Once more the choice is in our hands. We created the ideal points, and it is for us to say what they shall constitute.

We have seen that in a central projection a vanishing point projects into a point at infinity. In the projection of Fig. 5, the vanishing points in the plane p project into the points at infinity in the plane p'. The vanishing points in p constitute the vanishing line V, and every line other than V projects into a line. Consequently, it serves our purpose to agree that the points at infinity in the plane p' constitute a line.

AGREEMENT 2. *The ideal points in a plane shall constitute an ideal straight line, the line at infinity in the plane.*

To distinguish the points and lines originally at our disposal from the newly created ones, we shall call them *finite* points and lines. Similarly, we shall call the original plane *the finite plane.* The new plane, obtained by adjoining to the finite points the ideal points, shall be known as *the extended plane.*

Extended Space. The extension of finite space by the adjunction of points at infinity is effected by the following agreements.

AGREEMENT 3. *Corresponding to each direction in finite space, there is created a point at infinity. This point shall be thought of as lying on every line in the given direction and on no other finite line.*

AGREEMENT 4. *The points at infinity in the directions perpendicular to a given direction shall constitute an ideal line, or line at infinity. This line, and the points on it, shall be thought of as lying in every plane perpendicular to the given direction and in no other finite plane.*

AGREEMENT 5. *All the points at infinity shall constitute an ideal plane, the plane at infinity.*

It is evident that these agreements are in keeping with the previous agreements for the extension of a finite plane which lies in the given space.

The Suppression of Exceptions. It was in the finite domain that exceptional cases arose. In the extended domain, these exceptions can all be suppressed in that they can be looked upon as merely particular instances of the general case.

For example, hitherto the case of parallel projection was exceptional to that of central projection; now, parallel projection is simply a par-

ticular case of central projection, occurring when the center of projection is an ideal point.

Again, a triangle now projects always into a triangle, provided merely that we agree to extend the definition of a triangle to include the cases in which one or two of the vertices are points at infinity.

EXERCISES

1. Prove that a projection of a line L on a line L' establishes a one-to-one correspondence, *without exception*, between the points of the extended line L and those of the extended line L', (*a*) when there are vanishing points on L and L'; (*b*) when there are no vanishing points on L and L'.

2. Discuss the corresponding questions in the case of the projection of a plane upon a plane.

4. The Triangle Theorem of Desargues.* We are now in a position to prove one of the oldest and most important theorems of projective geometry.

Let ABC and $A'B'C'$ be two triangles in correspondence so that the vertices A and A', B and B', C and C' correspond; likewise, the sides BC and $B'C'$, CA and $C'A'$, AB and $A'B'$. Assume that each two corresponding sides and each two corresponding vertices are distinct.

DESARGUES' TRIANGLE THEOREM. *If the lines joining corresponding vertices of the two triangles are concurrent, the corresponding sides intersect in collinear points. Conversely, if corresponding sides intersect in collinear points, the lines joining corresponding vertices are concurrent.*†

We consider, first, the case in which the triangles ABC and $A'B'C'$ lie in distinct planes, p and p' (Fig. 6). If the lines joining corresponding vertices go through a point O, finite or ideal, then two corresponding sides lie in a plane through O, and hence intersect. The point of intersection, being a point in both the planes p and p', must lie on the line L in which these planes meet. Consequently, the pairs of corresponding sides intersect in points on L.

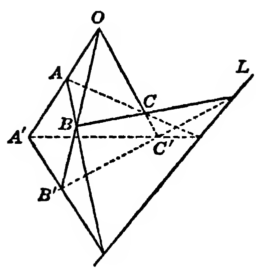


FIG. 6

* Desargues (1593–1662), a French architect, and a geometer of unusual ability, whose work was far ahead of his time. His triangle theorem appeared in a small book on conic sections, in 1639.

† In projective geometry, the sides of a triangle are not segments of straight lines, but complete straight lines. Hence there is no mention here of producing a side of the one triangle to meet a side of the other.

Conversely, if the pairs of corresponding sides of the two triangles intersect, then each two corresponding sides lie in a plane. The three planes thus obtained meet in a point O. On the other hand, the three lines in which the planes intersect, when taken in pairs, are evidently AA', BB', CC'. Hence AA', BB', CC' pass through O.

Let ABC and $A'B'C'$ now be two triangles so situated in the same plane p that corresponding sides intersect in points of a line L. Through these points in a second plane through L draw lines forming a third triangle $A''B''C''$, as shown in Fig. 7. Then ABC and $A''B''C''$ are triangles in different planes whose corresponding sides meet in collinear points. Hence the lines joining corresponding vertices, AA'', BB'', CC'', meet in a point V. Similarly, the lines $A'A''$, $B'B''$, $C'C''$ meet in a point V'. Inasmuch as the lines AA'' and $A'A''$ contain, respectively, the points V and V', the plane $AA'A''$ contains the line VV'. So also do the planes $BB'B''$ and $CC'C''$. Thus the planes $AA'A''$, $BB'B''$, $CC'C''$ pass through a line. Therefore the lines in which they intersect the plane p go through a point. But these lines are precisely AA', BB', CC', the lines joining corresponding vertices of the given triangles.

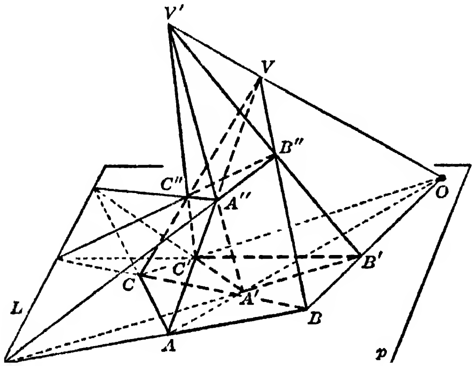


FIG. 7

Suppose, conversely, that the lines AA', BB', CC' go through a point, O. Choose two distinct points V, V' outside the plane p and collinear with O. Then, since the lines AA' and VV' intersect, in O, the lines AV and $A'V'$ intersect in a point A''. By similar reasoning BV and $B'V'$ intersect, in B'', and CV and $C'V'$ intersect, in C''. Inasmuch as the points $A''B''C''$ lie, respectively, on the lines VA, VB, VC, they are the vertices of a triangle which is related to the triangle ABC by projection from V. Similarly, this triangle $A''B''C''$ is related to the triangle $A'B'C'$ by projection from V'. Consequently, the sides of each of the given triangles must pass through the points in which the corresponding sides of the triangle $A''B''C''$ meet the

plane p. But these three points lie on a line, the line L in which the plane $A''B''C''$ intersects p. Thus, corresponding sides of the given triangles intersect on L.

Henceforth, unless the contrary is explicitly stated, we shall restrict ourselves to the geometry of the plane.

Exercise. The treatment of Desargues' Theorem in the text is in the extended domain. Give a complete statement of the theorem in the finite domain, taking care to enumerate all the exceptional cases. Select one of these exceptional cases, and show that it is actually covered by the treatment in the text.

5. Complete Quadrangles and Quadrilaterals.

Definition. *A complete quadrangle consists of four points, no three collinear, and the six lines which join them.*

The four points are called the *vertices*, and the six lines, the *sides*, of the complete quadrangle.

Definition. *A complete quadrilateral consists of four lines, no three concurrent, and the six points in which they intersect.*

The four lines are called the *sides*, and the six points, the *vertices*, of the complete quadrilateral.

By means of Desargues' Triangle Theorem we can deduce a number of projective theorems concerning complete quadrangles and quadrilaterals.

Theorem 1. *If the points of intersection of five pairs of corresponding* sides of two complete quadrangles lie on a line, the sides of the sixth pair intersect on this line, and the four lines joining corresponding vertices go through a point.*

Let the two quadrangles be $ABCD$ and $A'B'C'D'$, and let CD and $C'D'$ be the sixth pair of sides (Fig. 8). Since the corresponding sides of the triangles ABC and $A'B'C'$ intersect in points on the line L, the lines AA', BB', CC' are concurrent. Similarly, AA', BB', DD' are concurrent. It follows that AA', BB', CC', DD' all go through the same point,

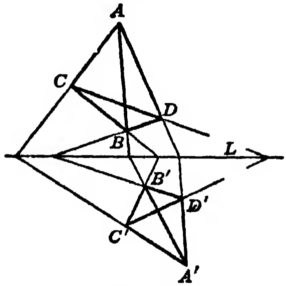


Fig. 8

* It is assumed that the two quadrangles are in a correspondence similar to that of the two triangles in Desargues' Theorem.

the point common to AA' and BB'.* Moreover, inasmuch as we now know that the lines joining corresponding vertices of the triangles ACD and $A'C'D'$ are concurrent, it follows that CD and $C'D'$ intersect on L.

The converse of the theorem is not true. We have, instead:

THEOREM 2. *If the lines joining corresponding vertices of two complete quadrangles are concurrent, the six points of intersection of the pairs of corresponding sides lie by threes on four lines and are then, in general, the vertices of a complete quadrilateral.*

The proof of the theorem is left to the reader.

6. The Principle of Duality. The principle which bears this name is of paramount importance in projective geometry.

DEFINITION 1. *Point and line are called dual elements.*

DEFINITION 2. *The drawing of a line through a point and the marking of a point on a line are known as dual operations.*

We should, then, say that the drawing of two lines through a point and the marking of two points on a line, or the bringing of two lines to intersect in a point and the joining of two points by a line, are also dual operations.

The two figures thus obtained, namely, the figure of two lines through a point and that of two points on a line, we shall call *dual figures.*

DEFINITION 3. *Two figures consisting of points and lines shall be called dual if one figure can be obtained from the other by replacing each element in it by the dual element and each operation by the dual operation.*

The dual of a complete quadrangle is a complete quadrilateral, and vice versa. A complete quadrangle is formed by taking four points, no three on a line, and joining them in pairs by lines. The dual figure is obtained by taking four lines, no three through a point, and bringing them to intersect in pairs, and is, therefore, a complete quadrilateral.

From Definition 2, it is clear that the figure of a line and a point on it is dual to itself, or *self-dual.*

* The reasoning here is invalid if AA' and BB' are the same line. However, since the reasoning is always valid as long as AA' and BB' are distinct lines, the conclusion is assured, by continuity, when AA' and BB' coincide. For, two fixed quadrangles so situated that AA' and BB' are identical can be considered as the limits of two variable quadrangles for which AA' and BB' are always distinct; and, if the conclusion in question can always be drawn for the variable quadrangles, it remains valid in the limit for the fixed quadrangles.

If we think of a triangle as formed by three noncollinear points and the lines joining them, the dual consists of three nonconcurrent lines and their points of intersection, a triangle. Thus a triangle is self-dual.

Let us now construct the dual of the figure representing Desargues' Theorem, and at the same time dualize the facts of the theorem itself. The duals of the two triangles are two new triangles. Dual to bringing two corresponding sides of the one pair of triangles to intersect in a point is the joining of two corresponding vertices of the other pair of triangles by a line. Consequently, the dual of the fact that corresponding sides of the one pair of triangles intersect in collinear points is the fact that the corresponding vertices of the other pair of triangles lie on concurrent lines. Hence, the hypothesis and the conclusion of the *direct* theorem of Desargues have, respectively, as their duals, the hypothesis and the conclusion of the *converse.* We express these facts by saying that the direct and converse theorems are dual theorems, or that Desargues' Theorem, as a whole, is self-dual.

Just as we have dualized Desargues' Theorem, so we can dualize many other theorems. Thus the dual of Theorem 1, § 5 is:

THEOREM 1. *If the lines joining five pairs of corresponding vertices of two complete quadrilaterals go through a point, the line joining the sixth pair goes through this point, and the four points of intersection of the corresponding sides lie on a line.*

Of course it still remains to show that this is a true theorem.

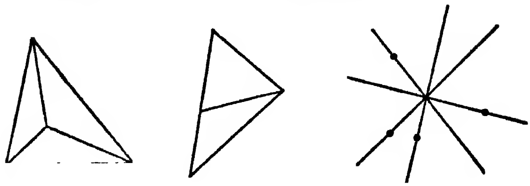
Remark. The figures and properties which we have been able to dualize, on the basis of our definitions, are all projective. Thus the principle of duality is peculiar to projective geometry. It has no analog in metric geometry. There is, for example, no possible basis for dualizing a circle, or a line-segment of length two, or an angle of 30°.*

EXERCISES

1. Describe and draw the dual of a complete five-point: five points, no three collinear and the lines which join them.

* The idea of duality was exploited by Poncelet in his systematic use of the theory of poles and polars, a theory in which dual figures correspond to one another. The first independent statement of the principle of duality was given by Gergonne in 1826.

2. Draw the figures dual to the following figures.



3. Prove Theorem 1.

4. State and prove the dual of Theorem 2, § 5.

5. Two vertices of a complete quadrilateral which do not lie on a side of the quadrilateral are called *opposite vertices*, and the line joining them is known as a *diagonal*. What are the dual definitions, of *opposite sides* and *diagonal points*, for a complete quadrangle? Draw figures.

6. If the vertices of a complete quadrangle lie on the sides of a complete quadrilateral so that two opposite sides of the quadrangle are concurrent with two diagonals of the quadrilateral, then two opposite vertices of the quadrilateral are collinear with two diagonal points of the quadrangle. Prove this theorem and state the dual. How is the dual related to the theorem?

CHAPTER III

HOMOGENEOUS CARTESIAN COORDINATES. LINEAR DEPENDENCE OF POINTS AND LINES

1. Homogeneous Cartesian Coordinates. In order to take full advantage of the points at infinity, we need coordinates for them. These are not to be obtained in the usual system of Cartesian coordinates, since every pair of numbers (x, y) has already been assigned as coordinates to a finite point. We are thus led to introduce a new coordinate system.

The new coordinates we shall call *homogeneous* (Cartesian) coordinates. We define them first for finite points in terms of the old, or *nonhomogeneous* (Cartesian) coordinates.

DEFINITION. *Homogeneous coordinates* (x_1, x_2, x_3) *of the finite point* (x, y) *are any three numbers* x_1, x_2, x_3 *for which*

$$\frac{x_1}{x_3} = x, \qquad \frac{x_2}{x_3} = y.$$

Sets of homogeneous coordinates of (x, y) are $(x, y, 1)$, $(-3x, -3y, -3)$, and (rx, ry, r), where r is any number, not zero.

A finite point has, then, infinitely many sets of homogeneous coordinates; each two sets are proportional, and the third coordinate of every set is different from zero. Conversely, any three numbers, the third of which is not zero, are homogeneous coordinates of a definite finite point, namely, the point $x = x_1/x_3$, $y = x_2/x_3$. Thus $(3, -2, 4)$ are homogeneous coordinates of the point $(3/4, -1/2)$.

Corresponding, therefore, to the finite points, we have as coordinates all the number triples (x_1, x_2, x_3) for which $x_3 \neq 0$. There remain the triples $(x_1, x_2, 0)$. We hope to be able to assign them as coordinates to the points at infinity.

Suppose we fix our attention on the point at infinity in the direction whose slope is λ. An arbitrary line L through this point, that is, an arbitrary line of slope λ, has the equation

$$y = \lambda x + b.$$

Let the variable point P: (x, y) recede indefinitely on L in either direction. If, then, homogeneous coordinates of P can be found which have limits which are independent of b, that is, independent of the particular line of slope λ chosen, these limits are reasonable homogeneous coordinates to assign to the point at infinity.

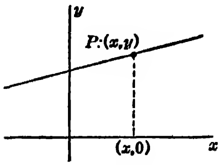


FIG. 1

Nonhomogeneous coordinates of P are $(x, \lambda x + b)$. When x becomes positively infinite, P recedes indefinitely on L in the one direction, and when x becomes negatively infinite, P recedes indefinitely in the other direction.

A set of homogeneous coordinates of P is $(x, \lambda x + b, 1)$. Here the first two coordinates become infinite with x. Consider, however, the set $(1, \lambda + (b/x), 1/x)$. When x becomes infinite, positively or negatively, these coordinates actually have limits $(1, \lambda, 0)$ which are independent of b.

Accordingly, we assign to the point at infinity in the direction of slope λ the sets of homogeneous coordinates $(r, r\lambda, 0)$, where r is any number, not zero.

DEFINITION. *Any number triple* $(x_1, x_2, 0)$ *for which*

$$\frac{x_2}{x_1} = \lambda$$

constitutes a set of homogeneous coordinates of the point at infinity in the direction of slope λ.*

We have now given to all the points of the extended plane homogeneous coordinates, and have thereby exhausted all triples save one: $(0, 0, 0)$. We agree not to use this triple at all.†

Equation in Homogeneous Coordinates of a Straight Line. If, in the equation

(1) $$a_1x + a_2y + a_3 = 0$$

of a straight line L, we set for x and y

$$x = \frac{x_1}{x_3}, \qquad y = \frac{x_2}{x_3}$$

* This definition and the previous discussion do not cover the point at infinity in the direction of the y-axis. We leave this case to the student.

† Homogeneous Cartesian coordinates were introduced in 1835 by the German geometer and physicist, Pluecker (1801–68). The line at infinity and points at infinity had already been employed systematically by Poncelet.

and multiply by x_3, we obtain the equation

$$(2) \qquad a_1x_1 + a_2x_2 + a_3x_3 = 0.$$

Equation (2) is the equation in homogeneous coordinates of the line L. For, it is evidently satisfied by the homogeneous coordinates of those and only those finite points which lie on L, and the coordinates of an ideal point P satisfy it if and only if P is the point at infinity on L; see Ex. 4.

Since the point (x_1, x_2, x_3) is an ideal point if and only if $x_3 = 0$ the equation

$$(3) \qquad x_3 = 0$$

is the equation of the line at infinity.

Equation (1) represents a straight line only if a_1 and a_2 are not both zero. But equation (2) always represents a straight line, provided only that a_1, a_2, a_3 are not all zero. In particular, if $a_1 = a_2 = 0$, the line is the ideal line.

Equation (2) is not merely linear in x_1, x_2, x_3; it is also *homogeneous* in x_1, x_2, x_3. It is from this fact that the homogeneous coordinates take their name.

EXERCISES

1. Find the homogeneous coordinates of the following points; first give all sets and then choose a simple set.

(*a*) $(0, 0)$; (*b*) $(-2, 3)$; (*c*) $(\frac{2}{3}, \frac{3}{5})$; (*d*) $(0, 1)$;

(*e*) the point at infinity in the direction of slope 3/4;

(*f*) the point at infinity in the direction of the y-axis.

2. Identify the following points, giving the nonhomogeneous coordinates when they exist.

(*a*) $(2, 4, -1)$; (*b*) $(3, 4, 2)$; (*c*) $(2, 1, 0)$; (*d*) $(0, 1, 0)$.

3. What does each of the following equations represent?

(*a*) $x_1 + x_2 - 4x_3 = 0$; (*b*) $x_1 + 2x_2 = 0$; (*c*) $x_2 - 3x_3 = 0$.

4. Show that the coordinates of an ideal point P satisfy equation (2) if and only if the point is the point at infinity on the line L.

2. Applications to Points and Straight Lines.

THEOREM 1 *a*. *The two points* (a_1, a_2, a_3), (b_1, b_2, b_3) *are identical if and only if*

$$\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} = 0, \qquad \begin{vmatrix} a_3 & a_1 \\ b_3 & b_1 \end{vmatrix} = 0, \qquad \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = 0.$$

Theorem 1 *b*. *The two straight lines,*

$$(1)\qquad \begin{aligned} a_1x_1 + a_2x_2 + a_3x_3 &= 0,\\ b_1x_1 + b_2x_2 + b_3x_3 &= 0, \end{aligned}$$

are identical if and only if

$$\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} = 0, \qquad \begin{vmatrix} a_3 & a_1 \\ b_3 & b_1 \end{vmatrix} = 0, \qquad \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = 0.$$

These theorems are an immediate consequence of the fact that the two points, or the two lines, are the same when and only when the two number triples a_1, a_2, a_3 and b_1, b_2, b_3 are proportional; see Ch. I, § 3, Th. 2.

Theorem 2 *a*. *Homogeneous coordinates of the point of intersection of the two distinct lines* (1) *are*

$$(2)\qquad x_1 = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}, \qquad x_2 = \begin{vmatrix} a_3 & a_1 \\ b_3 & b_1 \end{vmatrix}, \qquad x_3 = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}.$$

For, by Ch. I, § 2, one simultaneous solution of equations (1) is given by (2), and every other solution is proportional to this solution.

The point of intersection (2) of the two lines (1) lies on a third line,

$$c_1x_1 + c_2x_2 + c_3x_3 = 0,$$

if and only if

$$\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} c_1 + \begin{vmatrix} a_3 & a_1 \\ b_3 & b_1 \end{vmatrix} c_2 + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} c_3 = 0,$$

or

$$|a\,b\,c| = 0.$$

Theorem 3 *a*. *Three lines are concurrent if and only if the determinant of the coefficients in their equations has the value zero.*

We give a second proof. The three lines have a point in common when and only when their three equations have a simultaneous solution other than $x_1 = 0$, $x_2 = 0$, $x_3 = 0$; (0, 0, 0) are not coordinates of a point. But a necessary and sufficient condition that the three equations have a solution other than 0, 0, 0 is that the determinant of the coefficients vanish.

Theorem 2 *b*. *An equation of the line joining the distinct points* (a_1, a_2, a_3) *and* (b_1, b_2, b_3) *is*

$$(3)\qquad |x\,a\,b| = 0.$$

An equation of the line must be of the form

$$r_1x_1 + r_2x_2 + r_3x_3 = 0.$$

Since (a_1, a_2, a_3) and (b_1, b_2, b_3) lie on the line,

$$a_1r_1 + a_2r_2 + a_3r_3 = 0,$$
$$b_1r_1 + b_2r_2 + b_3r_3 = 0.$$

These equations in r_1, r_2, r_3 are the same as equations (1) in x_1, x_2, x_3. Hence a set of values for r_1, r_2, r_3 is given by (2), and an equation of the line is

$$\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} x_1 + \begin{vmatrix} a_3 & a_1 \\ b_3 & b_1 \end{vmatrix} x_2 + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} x_3 = 0,$$

or

$$|x\,a\,b| = 0.$$

THEOREM 3 *b*. *Three points are collinear if and only if the determinant of their homogeneous coordinates vanishes.*

For, a third point (c_1, c_2, c_3) lies on the line (3) if and only if $|c\,a\,b| = 0$, or $|a\,b\,c| = 0$.*

Cyclic Order and Advancement. The determinants in (2) are the two-rowed determinants formed from the matrix

$$\begin{Vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{Vmatrix}.$$

The columns of the matrix appearing in the three determinants are respectively the columns 2 3, 3 1, 1 2.

We have here an example of what is known as cyclic order. If the numbers 1, 2, 3 are distributed on a circle (Fig. 2), and the circle is traversed in the direction of the arrow, the pairs of numbers occur precisely in the orders 2 3, 3 1, 1 2.

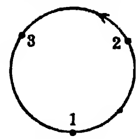


FIG. 2

When we proceed on the circle in the direction of the arrow, 1 advances to 2, 2 to 3, and 3 to 1. If, in the first of the equations (2), we advance all the subscripts cyclicly in this manner, the result is the second equation. Similarly, if the subscripts in the second equation are advanced cyclicly, the third equation is obtained. Hence, to write all three equations, we have but to know the first.

EXERCISES

1. Find the coordinates of the point of intersection of each of the following pairs of lines:

(*a*) $\quad 2x_1 - 3x_2 + 4x_3 = 0, \qquad x_1 + x_2 + x_3 = 0;$

(*b*) $\quad 4x_1 + 6x_2 + x_3 = 0, \qquad 2x_1 + 3x_2 - 2x_3 = 0.$

* The proof assumes that at least two of the points are distinct. The theorem is true, however, if all three points coincide; for then, on the one hand, $|a\,b\,c| = 0$, and, on the other, there is a line through the three points. Similar remarks apply to the first proof of Theorem 3 *a*.

2. Find the equation of the line joining the points

(*a*) $(1, -1, 2)$, $(0, 1, 4)$; (*b*) $(4, 1, -2)$, $(1, 1, 0)$.

3. Are the lines

$$x_1 - x_2 - x_3 = 0, \qquad 2x_1 + x_2 + 3x_3 = 0, \qquad 7x_1 - x_2 + 3x_3 = 0$$

concurrent? If so, find their common point.

4. Are the points $(2, 3, 1)$, $(5, -2, 2)$, $(1, -8, 0)$ collinear?

5. Check analytically the fact that two finite lines intersect in a point at infinity if and only if they are parallel.

6. Deduce from Th. 2 *b* a determinant form of the equation in nonhomogeneous coordinates of the line joining the distinct points (x_1, y_1), (x_2, y_2).

7. Prove that two *finite* points (a_1, a_2, a_3), (b_1, b_2, b_3) are identical if and only if $a_2b_3 - a_3b_2 = 0$, $a_3b_1 - a_1b_3 = 0$.

3. Abridged Notation. In the calculus, it is common practice to denote functions by simple symbols, for example, the function $x^2 + y^2 - 1$ of the two variables x, y by $f(x, y)$:

$$f(x, y) \equiv x^2 + y^2 - 1.$$

It is convenient to carry over this practice to geometry. Here we are interested not so much in the function as in the locus of the equation obtained by equating the function to zero. Thus, $f(x, y) = 0$, or, more briefly, $f = 0$, is a short or abridged notation for the equation

$$x^2 + y^2 - 1 = 0$$

of the unit circle.

Similarly, $\alpha(x_1, x_2, x_3) = 0$, or $\alpha = 0$, where

$$\alpha(x_1, x_2, x_3) \equiv a_1x_1 + a_2x_2 + a_3x_3,$$

is the equation in homogeneous coordinates of a straight line.

The relation $\rho(x_0, y_0) = 0$ expresses the fact that the value of the function $\rho(x, y)$ for $x = x_0$, $y = y_0$ is zero, or that the point (x_0, y_0) lies on the locus whose equation is $\rho(x, y) = 0$. Thus, $f(1, 0) = 0$ expresses the fact that the point $(1, 0)$ lies on the unit circle $f(x, y) = 0$.*

4. Linear Combination of Two Straight Lines. Pencils of Lines. Let

$$\alpha = 0, \qquad \alpha(x_1, x_2, x_3) \equiv a_1x_1 + a_2x_2 + a_3x_3,$$
$$\beta = 0, \qquad \beta(x_1, x_2, x_3) \equiv b_1x_1 + b_2x_2 + b_3x_3,$$

be two distinct straight lines and form the equation

$$k\alpha + l\beta = 0,$$

* The method of abridged notation, though employed earlier, was a favorite with Pluecker and came into prominence through his systematic use of it.

that is,

$$(k a_1 + l b_1)x_1 + (k a_2 + l b_2)x_2 + (k a_3 + l b_3)x_3 = 0,$$

where k and l are constants, not both zero.

The coefficients of x_1, x_2, x_3 in this equation cannot all vanish; for, if they were all zero, the number triples a_1, a_2, a_3 and b_1, b_2, b_3 would be proportional (Ch. I, § 3, Th. 2) and the lines $\alpha = 0$ and $\beta = 0$ would not be distinct. The equation represents, therefore, a straight line. This line is called *a linear combination of the given lines.*

THEOREM 1. *A linear combination of two distinct straight lines is a straight line passing through their point of intersection.*

If (r_1, r_2, r_3) is the point common to the lines $\alpha = 0$ and $\beta = 0$,

$$\alpha(r_1, r_2, r_3) = 0, \qquad \beta(r_1, r_2, r_3) = 0.$$

Therefore, no matter what values k and l have,

$$k\,\alpha(r_1, r_2, r_3) + l\,\beta(r_1, r_2, r_3) = 0.$$

Hence (r_1, r_2, r_3) also lies on the line $k\,\alpha + l\,\beta = 0$.

THEOREM 2. *Any straight line passing through the point of intersection of two distinct lines can be expressed as a linear combination of the two lines.*

Let L be the given line through the point P common to $\alpha = 0$ and $\beta = 0$, and let (s_1, s_2, s_3) be a point of L other than P. Since the line $k\,\alpha + l\,\beta = 0$ goes through P, it will coincide with L if it contains the point (s_1, s_2, s_3), that is, if

$$k\,\alpha(s_1, s_2, s_3) + l\,\beta(s_1, s_2, s_3) = 0.$$

This is an equation in k and l; $\alpha(s_1, s_2, s_3)$ and $\beta(s_1, s_2, s_3)$ are constants. One solution is $k = \beta(s_1, s_2, s_3)$, $l = -\alpha(s_1, s_2, s_3)$; and for these values of k and l, $k\,\alpha + l\,\beta = 0$ represents L.

Example. Let it be required to find the equation of the line joining the point (2, 1, 1) to the point of intersection of the lines

$$2\,x_1 - 3\,x_2 + x_3 = 0, \qquad 5\,x_1 - 3\,x_2 - 2\,x_3 = 0.$$

The linear combination of the given lines,

$$k\,(2\,x_1 - 3\,x_2 + x_3) + l\,(5\,x_1 - 3\,x_2 - 2\,x_3) = 0,$$

contains the point (2, 1, 1) if $2\,k + 5\,l = 0$, or $k = 5$, $l = -2$. Thus the required line is found to have the equation $x_2 - x_3 = 0$.

Pencils of Lines. The totality of lines through a point, finite or ideal, is known as a *pencil of lines.* The point is called the *vertex* of the pencil.

From Theorems 1 and 2 we have:

THEOREM 3. *If $\alpha = 0$ and $\beta = 0$ are distinct straight lines, the totality of lines represented by the equation $k\alpha + l\beta = 0$, when k, l take on all possible pairs of values other than* 0, 0, *is the pencil of lines having the point of intersection of $\alpha = 0$ and $\beta = 0$ as vertex.*

EXERCISES

1. Find the equation of the straight line which passes through the origin and the point of intersection of the lines

$$3x + 2y + 1 = 0, \qquad 4x - y - 2 = 0.$$

Do twice, once using homogeneous coordinates and once using nonhomogeneous coordinates.*

2. Find the equation of the straight line which passes through the point of intersection of the lines

$$2x_1 + 5x_2 - x_3 = 0, \qquad 3x_1 + 2x_2 - 4x_3 = 0$$

and is parallel to the axis of x.

3. Find the equation of the straight line which passes through the point of intersection of the lines of Ex. 2, and is perpendicular to the first of the lines in Ex. 1.

4. What linear combination of the two parallel lines

$$6x_1 - 9x_2 + 2x_3 = 0, \qquad 4x_1 - 6x_2 - 3x_3 = 0$$

is the line at infinity?

5. Find the equation of the straight line parallel to the lines of Ex. 4, and going through the point $(2, 3, -2)$.

6. If $\alpha = 0$ is a finite line and $\beta = 0$ is the line at infinity, what can you say of the pencil of lines $k\alpha + l\beta = 0$?

7. What does each of the following equations represent, if k and l are arbitrary constants?

$$(a) \quad k(3x_1 + 5x_3) + lx_3 = 0; \qquad (b) \quad kx - ly = 0.$$

5. Linear Dependence of Straight Lines. If a function $f(x_1, x_2, x_3)$ of x_1, x_2, x_3 has the value zero for all sets of values of x_1, x_2, x_3, the function is said to be identically zero: $f(x_1, x_2, x_3) \equiv 0$.

* The theory of linear combination of two straight lines remains essentially the same when the homogeneous coordinates are replaced by nonhomogeneous coordinates; see *Analytic Geometry*, Ch. IX, § 3.

Lemma. *If* $A_1x_1 + A_2x_2 + A_3x_3$ *vanishes identically:*

$$A_1x_1 + A_2x_2 + A_3x_3 \equiv 0,$$

then $A_1 = 0,\ A_2 = 0,\ A_3 = 0.$

For, since $A_1x_1 + A_2x_2 + A_3x_3$ vanishes for all sets of values of x_1, x_2, x_3, it vanishes for $x_1 = 1$, $x_2 = 0$, $x_3 = 0$; hence $A_1 = 0$. Similarly, $A_2 = 0$, and $A_3 = 0$.

Definition. *The two lines*

(1) $\alpha \equiv a_1x_1 + a_2x_2 + a_3x_3 = 0, \qquad \beta \equiv b_1x_1 + b_2x_2 + b_3x_3 = 0$

are linearly dependent if and only if two constants k, l, *not both zero, exist so that* $k\alpha + l\beta$ *vanishes for all sets of values of* x_1, x_2, x_3*:*

$$k\alpha + l\beta \equiv 0. \tag{2}$$

Since

$$k\alpha + l\beta \equiv (k a_1 + l b_1)x_1 + (k a_2 + l b_2)x_2 + (k a_3 + l b_3)x_3,$$

it follows that $k\alpha + l\beta$ vanishes identically if and only if

$$k a_1 + l b_1 = 0, \qquad k a_2 + l b_2 = 0, \qquad k a_3 + l b_3 = 0. \tag{3}$$

The existence of k, l, not both zero, so that equations (3) are satisfied is precisely the condition that the sets of coefficients a_1, a_2, a_3 and b_1, b_2, b_3 in the equations of the lines be linearly dependent; see Ch. I, § 3. But then the lines are identical, and conversely.

Theorem 1. *Two lines are linearly dependent if and only if they are identical.*

The definition of linear dependence for any number of lines is similar to that for two lines. Thus, the three lines,

$$\begin{aligned} \alpha &\equiv a_1x_1 + a_2x_2 + a_3x_3 = 0,\\ \beta &\equiv b_1x_1 + b_2x_2 + b_3x_3 = 0,\\ \gamma &\equiv c_1x_1 + c_2x_2 + c_3x_3 = 0, \end{aligned} \tag{4}$$

are linearly dependent, if and only if three constants k, l, m, not all zero, exist so that

$$k\alpha + l\beta + m\gamma \equiv 0, \tag{5}$$

or, what is the same thing, so that

$$\begin{aligned} k a_1 + l b_1 + m c_1 &= 0,\\ k a_2 + l b_2 + m c_2 &= 0,\\ k a_3 + l b_3 + m c_3 &= 0. \end{aligned} \tag{6}$$

Equations (6) have a solution for k, l, m, other than 0, 0, 0, if and only if $|a\,b\,c| = 0$. By § 2, Th. 3 a, this is precisely a condition necessary and sufficient that the lines (4) be concurrent.

THEOREM 2. *Three lines are linearly dependent if and only if they are concurrent.*

We give a second proof, based on identities. Assume that k, l, m, not all zero, exist so that the identity (5) holds, and suppose, for example, that $m \neq 0$. Then

$$\gamma \equiv -\frac{k}{m}\alpha - \frac{l}{m}\beta.$$

This identity states that the line $\gamma = 0$ is the same line as

$$-\frac{k}{m}\alpha - \frac{l}{m}\beta = 0, \qquad \text{or} \qquad k\alpha + l\beta = 0.$$

Hence $\gamma = 0$ passes through the point of intersection of $\alpha = 0$ and $\beta = 0$, if they are distinct, and is identical with them if they coincide.

Conversely, let $\alpha = 0$, $\beta = 0$, $\gamma = 0$ be concurrent. If $\alpha = 0$ and $\beta = 0$ are distinct lines, then, by § 4, Th. 2, $\gamma = 0$ must be a linear combination of them:

$$\gamma \equiv k\alpha + l\beta.$$

Hence

$$k\alpha + l\beta - \gamma \equiv 0,$$

and the lines are linearly dependent. On the other hand, if the lines $\alpha = 0$ and $\beta = 0$ are identical, there exist, according to Th. 1, two constants, k, l, not both zero, so that $k\alpha + l\beta \equiv 0$. Consequently, the identity (5), where $m = 0$ and k and l have the values just noted, is valid, and the three lines are linearly dependent.

THEOREM 3. *If three distinct lines are linearly dependent, equations of the lines can be so chosen that the constants of dependence can all be taken as unity.*

If $\alpha = 0$, $\beta = 0$, $\gamma = 0$ are three *distinct* lines which are linearly dependent: $k\alpha + l\beta + m\gamma \equiv 0$, no one of the constants of dependence can be zero. Then $k\alpha = 0$, $l\beta = 0$, $m\gamma = 0$ are also equations of the lines and when we set $k\alpha \equiv \alpha'$, $l\beta \equiv \beta'$, $m\gamma \equiv \gamma'$, the identity becomes $\alpha' + \beta' + \gamma' \equiv 0$. Thus the theorem is proved.

THEOREM 4. *The linear dependence of a number of lines is coextensive with the linear dependence of the sets of coefficients in the equations of the lines.*

The truth of the theorem in the case of two lines has already been noted in the proof of Theorem 1. Its verification in the case of three or more lines is left to the reader.

EXERCISES

1. Show that the lines

$$4x_1 - 2x_2 + 6x_3 = 0, \qquad 6x_1 - 3x_2 + 9x_3 = 0$$

are linearly dependent and find values for the constants of dependence.

2. The same for the lines

$$5x_1 + 6x_2 - 3x_3 = 0, \qquad 2x_1 + 3x_2 - 4x_3 = 0, \qquad 4x_1 + 3x_2 + 6x_3 = 0.$$

3. Choose equations for the lines in Ex. 2 so that the constants of dependence in the identity expressing linear dependence can all be taken equal to unity.

4. Prove that four lines are always linearly dependent.

5. Show that, if no three of four given lines are concurrent, equations $\alpha = 0$, $\beta = 0$, $\gamma = 0$, $\delta = 0$ for the four lines can be so chosen that $\alpha + \beta + \gamma + \delta \equiv 0$.

6. Show that two finite lines are parallel if and only if they and the line at infinity are linearly dependent. Hence prove that the finite lines

$$\alpha \equiv a_1x + a_2y + a_3 = 0, \qquad \beta \equiv b_1x + b_2y + b_3 = 0$$

are parallel, but not identical, if and only if constants k, l, m, $m \neq 0$, exist so that $k\alpha + l\beta \equiv m$.

6. Linear Dependence of Points. Ranges of Points. For the sake of brevity, we shall henceforth denote the point (x_1, x_2, x_3) symbolically by x and write x: (x_1, x_2, x_3).

Definition. *A number of points are said to be linearly dependent if and only if their sets of homogeneous coordinates are linearly dependent.*

In the case of two points, the definition requires that the coordinates of the two points be proportional. Hence:

Theorem 1. *Two points are linearly dependent when and only when they are identical.*

The three points a: (a_1, a_2, a_3), b: (b_1, b_2, b_3), c: (c_1, c_2, c_3) are linearly dependent if and only if k, l, m, not all zero, exist so that the equations,

$$\begin{aligned} k a_1 + l b_1 + m c_1 &= 0, \\ k a_2 + l b_2 + m c_2 &= 0, \\ k a_3 + l b_3 + m c_3 &= 0, \end{aligned} \tag{1}$$

are satisfied. The condition, necessary and sufficient, for linear dependence is, therefore, that $|a\,b\,c| = 0$. But, by § 2, Th. 3 b, this is the condition that the three points be collinear.

Theorem 2. *Three points are linearly dependent if and only if they are collinear.*

It will be noted that these theorems are the precise duals of the corresponding theorems concerning the linear dependence of lines.

Equations (1) can be written more compactly in the form

$$k\,a_i + l\,b_i + m\,c_i = 0, \qquad (i = 1, 2, 3).$$

We go still further and drop the subscripts altogether, writing

$$k\,a + l\,b + m\,c = 0$$

as a *symbolic equation* standing for all three equations.

Ranges of Points. A range of points is the totality of points on a straight line. It is the dual of a pencil of lines. Since a pencil of lines consists of all the lines which are linear combinations of two distinct lines, a range of points may be expected to consist of all the points which are linear combinations of two distinct points. This is indeed the case.

Theorem 3. *If a and b are distinct points, the totality of points*

$$k\,a + l\,b\colon\ (k\,a_1 + l\,b_1,\ k\,a_2 + l\,b_2,\ k\,a_3 + l\,b_3),$$

where k, l take on all possible pairs of values other than 0, 0, *is the range of points on the line of a and b.*

To show that an arbitrary point $k\,a + l\,b$ lies on the line of a and b, denote the coordinates $k\,a_1 + l\,b_1$, $k\,a_2 + l\,b_2$, $k\,a_3 + l\,b_3$ of this point by c_1, c_2, c_3. Then

$$c_i = k\,a_i + l\,b_i, \qquad (i = 1, 2, 3),$$

or, symbolically,

$$k\,a + l\,b - c = 0.$$

Consequently, the points a, b, c are linearly dependent and c, or $k\,a + l\,b$, lies on the line of a and b.

Conversely, if a point c lies on the line of a and b, the points a, b, c are linearly dependent:

$$k\,a + l\,b + m\,c = 0.$$

Since a and b are distinct points, the constant m cannot be zero. Therefore we can divide by m, obtaining c:

$$c = -\frac{k}{m}\,a - \frac{l}{m}\,b,$$

as a linear combination of a and b.

Example. Find the points in which the line joining the points $(-6, 0)$ and $(6, 4)$ meets the parabola

$$y^2 - x - 4 = 0.$$

Homogeneous coordinates of the given points are $(-6, 0, 1)$, $(6, 4, 1)$ and those of an arbitrary point of their line are

$$(-6k + 6l, 4l, k + l).$$

These coordinates satisfy the equation in homogeneous coordinates of the parabola:

$$x_2^2 - x_1x_3 - 4x_3^2 = 0$$

if

$$16l^2 + 6(k^2 - l^2) - 4(k + l)^2 = 0,$$

or

$$k^2 - 4kl + 3l^2 = 0.$$

Thus $k - l = 0$ or $k - 3l = 0$. Setting $k = 1$, $l = 1$, and then $k = 3$, $l = 1$, we obtain as the coordinates of the required points $(0, 2, 1)$ and $(-3, 1, 1)$, or $(0, 2)$ and $(-3, 1)$.

EXERCISES

1. The three points $(2, 3, -2)$, $(4, 5, 2)$, $(1, 2, -4)$ are linearly dependent. Find values for the constants of dependence.

2. Choose coordinates for the points in Ex. 1 so that the constants of dependence can all be taken equal to unity.

3. What linear combination of the two points $(1, 1, 2)$, $(2, -1, 3)$ is the point at infinity on their line?

Find the coordinates of the point (or points) in which the line joining the given points meets the given curve.

	Given points	*Given curve*
4.	$(2, 1, 3)$, $(3, 1, 2)$,	$x_1 - x_2 + x_3 = 0$.
5.	$(5, 1, 5)$, $(1, 3, 1)$,	$2x^2 + y^2 - 3 = 0$.
6.	$(1, 0)$, $(0, -1)$,	$x^2 + y^2 - 2x - 1 = 0$.

7. Show that, if A, B, C are distinct collinear points, homogeneous coordinates a and b of A and B can be chosen so that C has the coordinates $a + b$.

7. Analytic Proof of Desargues' Triangle Theorem. To illustrate the usefulness of the ideas of linear combination and linear dependence, we give an analytic proof of Desargues' Theorem in the plane. We establish first:

THEOREM 1. *If the corresponding sides of the two triangles intersect in collinear points, the lines joining corresponding vertices are concurrent.*

Let the equations in homogeneous coordinates of the sides of the one triangle be $\alpha = 0$, $\beta = 0$, $\gamma = 0$, and those of the sides of the other triangle, $\alpha' = 0$, $\beta' = 0$, $\gamma' = 0$, as shown in the figure. Let the equation of the line L on which lie the points of intersection A'', B'', C'' of the corresponding sides be $\delta = 0$.

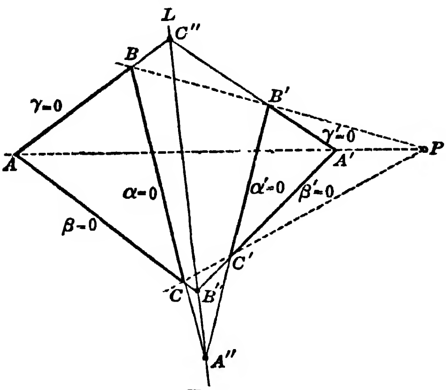


Fig. 3

Since L goes through the point A'' common to $\alpha = 0$ and $\alpha' = 0$, δ is identically the same as a certain linear combination of α and α': $\delta \equiv A\alpha - A'\alpha'$. Since L also goes through B'' and C'', we can, by similar reasoning, write $\delta \equiv B\beta - B'\beta'$ and $\delta \equiv C\gamma - C'\gamma'$. Hence *

$$A\alpha - A'\alpha' \equiv B\beta - B'\beta' \equiv C\gamma - C'\gamma' \equiv \delta. \tag{1}$$

From this continued identity, we obtain the three identities:

$$\begin{aligned} B\beta - C\gamma &\equiv B'\beta' - C'\gamma', \\ C\gamma - A\alpha &\equiv C'\gamma' - A'\alpha', \\ A\alpha - B\beta &\equiv A'\alpha' - B'\beta'. \end{aligned} \tag{2}$$

The first of these says that the equations $B\beta - C\gamma = 0$ and $B'\beta' - C'\gamma' = 0$ † represent the same line. This line is a linear combination of $\beta = 0$ and $\gamma = 0$, and hence goes through the point A; it is also a linear combination of $\beta' = 0$ and $\gamma' = 0$, and so goes through the point A'. Hence it is the line AA'.

Similarly, the second identity yields two equations for the line BB', and the third, two equations for CC'. Thus we have two sets of equations for the lines AA', BB', CC':

$$\begin{array}{llll} (3) & AA'\colon & B\beta - C\gamma = 0, & B'\beta' - C'\gamma' = 0, \\ & BB'\colon & C\gamma - A\alpha = 0, & C'\gamma' - A'\alpha' = 0, \\ & CC'\colon & A\alpha - B\beta = 0, & A'\alpha' - B'\beta' = 0. \end{array}$$

* The constants of combination A, A', B, B', C, C' are not to be confused with the lettering of the vertices in the figure.

† Since it is impossible that $B = C = 0$ or $B' = C' = 0$, these equations actually represent lines.

The equations of either set are linearly dependent with constants of dependence all unity. Hence the lines AA', BB', CC' are concurrent.

THEOREM 2. *If the lines joining corresponding vertices of the two triangles are concurrent, the corresponding sides intersect in collinear points.*

Theorem 2 is the dual of Theorem 1. Moreover, the theory of linear combination and linear dependence of straight lines used in the proof of Theorem 1 has a precise dual in the corresponding theory for points. Consequently, we can give for Theorem 2 a proof which is the dual of that of Theorem 1.

If a: (a_1, a_2, a_3), b, c are coordinates of the vertices A, B, C of the first triangle, a', b', c' coordinates of the vertices A', B', C' of the second triangle, and d: (d_1, d_2, d_3) coordinates of the point P (Fig. 3), we have, symbolically,

$$Aa - A'a' = Bb - B'b' = Cc - C'c' = d.$$

Let the reader justify these relations, and then complete the proof.

EXERCISES

1. Complete the proof of Theorem 2.

2. LEMMA A. *A necessary and sufficient condition that three distinct lines, one through each vertex of the triangle whose sides are $\alpha = 0$, $\beta = 0$, $\gamma = 0$, be concurrent is that their equations can be put in the forms:*

$$(4) \qquad B\beta - C\gamma = 0, \qquad C\gamma - A\alpha = 0, \qquad A\alpha - B\beta = 0.$$

The condition is evidently sufficient; equations (4) represent three concurrent lines, one through each vertex of the triangle; see equations (3).

It remains to show that the condition is necessary. The three lines, since they are concurrent, are linearly dependent. By § 5, Th. 3, their equations

$$B\beta + C'\gamma = 0, \qquad C\gamma + A'\alpha = 0, \qquad A\alpha + B'\beta = 0,$$

can be so chosen that the constants of dependence can all be taken equal to unity:

$$(B\beta + C'\gamma) + (C\gamma + A'\alpha) + (A\alpha + B'\beta) \equiv 0.$$

Hence

$$(A + A')\,\alpha + (B + B')\,\beta + (C + C')\,\gamma \equiv 0.$$

To complete the proof we conclude that

$$A + A' = 0, \qquad B + B' = 0, \qquad C + C' = 0.$$

What is the justification for this conclusion?

3. Prove Theorem 2, by showing that the steps in the proof of Theorem 1 can be retraced. Begin by establishing equations (3) with the help of Lemma A.

4. LEMMA B. *The dual of Lemma A.* State and prove.

5. Prove Theorem 1 by retracing the steps in the proof of Theorem 2.

8. Linear Dependence of Four Points or of Four Lines. By Ch. I, § 3, Th. 4, four number triples are always linearly dependent. Hence:

THEOREM 1. *Four points are always linearly dependent.*

In seeking the geometric significance of this fact, we consider merely the general case in which at least three of the four points are linearly independent.* Let a, b, c, d be the four points, and let a, b, c be linearly independent. Then, in the symbolic equation

$$ka + lb + mc + nd = 0 \tag{1}$$

which expresses the linear dependence of the four points, the constant n cannot be zero. Consequently, we can divide by n and write:

$$d = Aa + Bb + Cc,$$

where $A = -k/n$, $B = -l/n$, $C = -m/n$. Thus, we have the coordinates of the point d expressed as a linear combination of those of the three points a, b, c. But d might have been any point in the plane. Hence:

THEOREM 2. *The homogeneous coordinates of an arbitrary point of the plane can be expressed as a linear combination of those of three non-collinear points.*

The result may be stated in another way: The totality of points which are linear combinations of three noncollinear points consists of all the points of the plane.

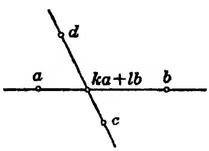


FIG. 4

Geometric proofs of Theorems 1 and 2 can be readily given. Consider the latter theorem. The point in which the line of a and b (Fig. 4) is met by the line of c and d † is a linear combination, $ka+lb$, of a and b, and the point d is a linear combination of this point, $ka + lb$, and c; that is, $d = A(ka + lb) + Cc$. Hence d is a linear combination of a, b, and c.

Theorems analogous to Theorems 1 and 2 hold for straight lines.

THEOREM 3. *Four straight lines are always linearly dependent.*

THEOREM 4. *The equation of an arbitrary line in the plane can be written as a linear combination of the equations of three nonconcurrent lines.*

* The special case in which each three of the four points are themselves linearly dependent presents nothing new: the four points are collinear.

† The theorem is obvious if d coincides with c.

EXERCISES

1. Express the point (1, 1, 1) as a linear combination of the three noncollinear points (1, 0, 0), (0, 1, 0), (0, 0, 1).

2. Give a geometric proof of Theorem 4.

3. Show that, if no three of four given points are collinear, coordinates a, b, c, d of the four given points can be so chosen that $a + b + c + d = 0$.

4. Determine specific values for the constants of dependence in (1) in the general case in which at least three of the four points are linearly independent.

9. Applications to Complete Quadrangles and Complete Quadrilaterals. Two sides of a complete quadrangle whose point of intersection is not a vertex of the quadrangle are called *opposite sides.* There are evidently three pairs of opposite sides: p_1, p_2; q_1, q_2; r_1, r_2 (Fig. 5). The points P, Q, R determined by the pairs of opposite sides are known as *diagonal points.*

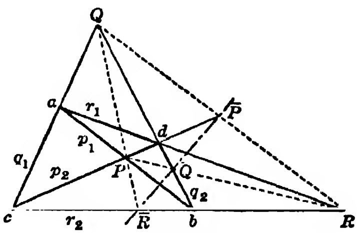


FIG. 5

According to § 8, Ex. 3, coordinates a, b, c, d of the vertices of the complete quadrangle can be chosen so that

$$a + b + c + d = 0. \tag{1}$$

We have then, for example,

$$a + b = -(c + d).$$

This symbolic equation says that $a + b$ and $c + d$ are coordinates of the same point. This point lies on the line p_1 of a and b, and also on the line p_2 of c and d; hence it must be the diagonal point P. In the same way we get two sets of coordinates for each of the other diagonal points. Tabulating the results, we have

$$\begin{array}{llll} P: & a + b & \text{or} & c + d \\ Q: & a + c & \text{or} & d + b \\ R: & a + d & \text{or} & b + c. \end{array} \tag{2}$$

Is it ever possible that the points P, Q, R be collinear? To answer this question, let us assume tentatively that the points are collinear. Their coordinates $c + d$, $d + b$, $b + c$ are then linearly dependent; constants B, C, D, not all zero, exist so that

$$B(c + d) + C(d + b) + D(b + c) = 0.$$

Hence

$$(C + D)\,b + (D + B)\,c + (B + C)\,d = 0.$$

Since B, C, D are not all zero, $C + D$, $D + B$, $B + C$ are not all zero. Hence the points b, c, d are linearly dependent. But this means that three vertices of the complete quadrangle are collinear, which is impossible.

THEOREM 1. *The diagonal points of a complete quadrangle are never collinear.*

The triangle whose vertices are the diagonal points is known as the *diagonal triangle* of the quadrangle.

EXERCISES

1. Show that the points $\bar{P}$, $\bar{Q}$, $\bar{R}$ in Fig. 5 have the coordinates $c - d$, $d - b$, $b - c$ and thus prove the theorem: The three points in which three nonconcurrent sides of a complete quadrangle, one through each diagonal point, meet the opposite sides of the diagonal triangle are collinear.

2. Give a geometric proof of this theorem.

3. Opposite vertices and diagonals of a complete quadrilateral are the duals of opposite sides and diagonal points of a complete quadrangle. Define them and prove that the diagonals are never concurrent.

4. State and prove the dual of the theorem in Ex. 1.

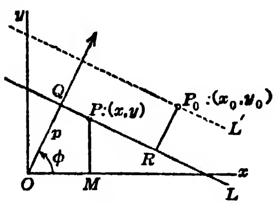


FIG. 6

10. Metric Applications. *Normal Form for the Equation of a Line.* Let L be a finite line which does not pass through the origin O of coordinates, and let Q be the foot of the perpendicular from O on L. The position of L is determined by the angle ϕ from the positive axis of x to the directed line OQ and the length $p > 0$ of the line-segment OQ. The equation of L in terms of ϕ and p is *

$$(1) \qquad x \cos \phi + y \sin \phi - p = 0.$$

Directed Distance from a Line to a Point. Let P_0 be an arbitrary point of the plane and let R be the foot of the perpendicular dropped from P_0 on the line L. The *directed* distance, d, from L to P_0 we define as the directed line-segment RP_0, taken positive or negative according as the direction from R to P_0 is the same as, or opposite to, that from O to Q. Evidently d is positive if P_0 lies on the opposite side of L from O, and negative if P_0 lies on the same side of L as O.

To obtain a formula for d, draw through P_0 a line L' parallel to

* The equation is obtained by equating the projection on OQ of the broken line OMP to the projection p of OP on OQ.

L. For L', $\phi' = \phi$ and $p' = p + d$, provided L' meets the half-line OQ, as in the figure. If L' meets the opposite half-line, $\phi' = \phi + \pi$ and $p' = -p - d$. In both cases the equation of L' is

$$x \cos \phi + y \sin \phi - p - d = 0.$$

Since P_0: (x_0, y_0) lies on L', we have

$$(2) \qquad d = x_0 \cos \phi + y_0 \sin \phi - p.$$

THEOREM 1. *When the equation of a line is given in normal form*

$$\alpha(x, y) \equiv x \cos \phi + y \sin \phi - p = 0,$$

the directed distance d *from the line to the point* (x_0, y_0) *is*

$$d = \alpha(x_0, y_0).$$

Let $\alpha = 0$ and $\beta = 0$ be the equations in normal form of two intersecting lines not passing through O. The linear combination $\alpha - k\beta = 0$ has, then, a simple geometric interpretation which can be readily established by means of Th. 1.

THEOREM 2. *The line* $\alpha - k\beta = 0$ *is the locus of a point which moves so that its directed distances from the lines* $\alpha = 0$ *and* $\beta = 0$ *are in the ratio* k. *If* $k > 0$, *the line passes through the two regions bounded by* $\alpha = 0$, $\beta = 0$, *one of which contains the origin; if* $k < 0$, *the line passes through the other two regions.*

COROLLARY. *The equations of the bisectors of the angles between* $\alpha = 0$ *and* $\beta = 0$ *are* $\alpha - \beta = 0$ *and* $\alpha + \beta = 0$.

Applications. We are now in a position to prove a number of theorems.

THEOREM 3. *The bisectors of the internal angles of a triangle meet in a point.*

Choose the origin O inside the triangle and let the equations in normal form of the sides be $\alpha = 0$, $\beta = 0$, $\gamma = 0$. The equations of the bisectors in question are, then,

$$\alpha - \beta = 0, \qquad \beta - \gamma = 0, \qquad \gamma - \alpha = 0.$$

But these equations are linearly dependent. Hence the bisectors are concurrent.

THEOREM 4. *The medians of a triangle are concurrent.*

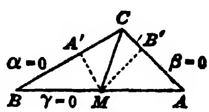


FIG. 7

Take O, as before, within the triangle. The distances MA' and MB' from the mid-point M of AB to the sides $\alpha = 0$ and $\beta = 0$

(Fig. 7) are

$$MA' = \tfrac{1}{2} c \sin B, \qquad MB' = \tfrac{1}{2} c \sin A,$$

where c is the length of the side AB. Hence the equation of the median CM is

$$\alpha \sin A - \beta \sin B = 0.$$

Similarly, the equations of the other medians are

$$\beta \sin B - \gamma \sin C = 0, \qquad \gamma \sin C - \alpha \sin A = 0.$$

The three medians are evidently linearly dependent, and therefore concurrent.

Application to Circles. Let the equations,

$$\alpha \equiv x^2 + y^2 + a_1 x + a_2 y + a_3 = 0,$$
$$\beta \equiv x^2 + y^2 + b_1 x + b_2 y + b_3 = 0,$$

represent two circles which intersect in two distinct points P_1 and P_2. Then the equation

$$k\alpha + l\beta = 0,$$

where k, l are constants not both zero, represents a curve which passes through the points P_1 and P_2. From the form of the equation:

$$(k + l)(x^2 + y^2) + (ka_1 + lb_1)x + (ka_2 + lb_2)y + (ka_3 + lb_3) = 0,$$

it is evident that this curve is a circle, unless $k + l = 0$.

In the exceptional case, the curve is a straight line. Since it contains the points P_1 and P_2, it must be the common chord P_1P_2 of the two circles. Its equation is evidently $\alpha - \beta = 0$.

It can be readily shown that, by choosing k, l suitably, $k\alpha + l\beta = 0$ can be made to represent any desired circle passing through P_1 and P_2. The totality of these circles, together with the common chord P_1P_2, is known as a *pencil* of circles.

Suppose now that

$$\gamma \equiv x^2 + y^2 + c_1 x + c_2 y + c_3 = 0$$

is a third circle which intersects each of the circles $\alpha = 0$ and $\beta = 0$ in distinct points. The common chords of the three circles, taken in pairs, have the equations

$$\alpha - \beta = 0, \qquad \beta - \gamma = 0, \qquad \gamma - \alpha = 0,$$

and are therefore concurrent.

Theorem 5. *If three mutually intersecting circles are taken in pairs and the common chords are drawn, the common chords are concurrent.**

* The point of concurrency may be a point at infinity.

Remark. It frequently happens that, in the proof of a theorem by abridged notation, the equations can be reinterpreted so that a new theorem results.

For example, suppose that in the proof of Theorem 3, in which $\alpha = 0$, $\beta = 0$, $\gamma = 0$ are the sides of a triangle and $\alpha - \beta = 0$, $\beta - \gamma = 0$, $\gamma - \alpha = 0$ the bisectors of the internal angles, we now think of $\alpha = 0$, $\beta = 0$, $\gamma = 0$ as representing circles. Then $\alpha - \beta = 0$, $\beta - \gamma = 0$, $\gamma - \alpha = 0$ are the common chords of the circles, and the proof of Theorem 3 becomes that of Theorem 5.

EXERCISES

1. Prove that

$$d = \frac{\alpha(x, y)}{\sqrt{a_1^2 + a_2^2}}$$

is a directed distance from the line

$$\alpha(x, y) \equiv a_1x + a_2y + a_3 = 0$$

to the point (x, y), positive for all points (x, y) on one side of the line and negative for all points (x, y) on the other side.

2. Prove that the bisectors of the exterior angles at two vertices of a triangle and the bisector of the interior angle at the third vertex are concurrent.

3. Show that the altitudes of a triangle are concurrent.

4. Prove that if $\alpha = 0$ and $\beta = 0$ are the equations in normal form of two intersecting lines, the lines $k\alpha - l\beta = 0$ and $l\alpha - k\beta = 0$ make the same angle with $\alpha - \beta = 0$.

5. Hence show that if three lines, one through each vertex of a triangle, meet in a point P_1, the three lines, one through each vertex, which make the same angles as the given lines with the bisectors of the angles, but lie on the opposite sides of the bisectors, meet in a point P_2. The two points P_1 and P_2 are known as *isogonal conjugate points* with respect to the given triangle.

6. Prove that, if two triangles correspond, vertex for vertex, so that the perpendiculars from the vertices of one on the opposite sides of the other are concurrent, then the perpendiculars from the vertices of the second on the opposite sides of the first are also concurrent.

7. Given three mutually intersecting circles and a point P. Show that the three circles which are determined by P and points of intersection of the pairs of given circles have in general a second point in common.

8. If four mutually intersecting circles can be paired in one way so that the points of intersection of the pairs of circles lie on a circle, show that the circles have this property no matter how they are paired.

9. What theorem concerning circles results from the proof of Desargues' Triangle Theorem when the equations in the proof are thought of as representing circles?

CHAPTER IV

HARMONIC DIVISION

1. Division of a Line-Segment. In the elementary theory of the division of a line-segment, there are in general two points which divide the segment in a given ratio, one external, and the other internal, to the segment.* For the purpose of this and later chapters, it is advisable to revise the theory so that but one point will correspond to a given ratio of division. This end is attained by replacing the absolute ratio hitherto used by an algebraic ratio, defined in terms of *directed* line-segments.

DEFINITION. *The algebraic ratio, μ, in which P divides the finite line-segment P_1P_2 is defined as the quotient of the directed line-segments $\overline{PP}_1$, $\overline{PP}_2$:*

$$\mu = \frac{\overline{PP}_1}{\overline{PP}_2}.$$

Inspection shows that, *according as P is external or internal to P_1P_2, μ is positive or negative.*

Since $\overline{PP}_1 = \mu\,\overline{PP}_2$, the projection of $\overline{PP}_1$ on an axis of coordinates is μ times the projection of $\overline{PP}_2$ on the axis. Hence we derive formulas for the coordinates of P.

THEOREM 1. *The coordinates (x, y) of the point P which divides the line-segment bounded by $P_1 : (x_1, y_1)$ and $P_2 : (x_2, y_2)$ in the ratio μ, $\neq 1$, are*

$$(1) \qquad x = \frac{x_1 - \mu x_2}{1 - \mu}, \qquad y = \frac{y_1 - \mu y_2}{1 - \mu}.$$

When $\mu = 1$, formulas (1) have no meaning. It is evident geometrically, however, that when μ approaches 1 as a limit, the point P recedes indefinitely. Moreover, in the homogeneous coordinates $(x_1 - \mu\,x_2,\ y_1 - \mu\,y_2,\ 1 - \mu)$ of P, we *can* set $\mu = 1$ and the result is the point at infinity on the line. Accordingly, we say that *the point at infinity on the line divides the line-segment P_1P_2 in the ratio $\mu = 1$.*

There is now a *unique* point on a finite line which divides a finite segment on the line in a given (algebraic) ratio.

* See *Analytic Geometry*, p. 17

We next introduce homogeneous coordinates and prove the theorem:

THEOREM 2. *The ratio, μ, in which the point $a + \lambda b$ divides the segment bounded by the finite points $a : (a_1, a_2, a_3)$ and $b : (b_1, b_2, b_3)$ is*

$$\mu = -\lambda \frac{b_3}{a_3}. \tag{2}$$

If $a + \lambda b$ is the point at infinity on the line of a and b, then $a_3 + \lambda b_3 = 0$ and hence $-\lambda b_3/a_3$ has the value 1, as prescribed by the theorem.

If $a + \lambda b$ is a finite point, μ can be determined from equations (1). These equations, when we set

$$x_1 = \frac{a_1}{a_3}, \qquad x_2 = \frac{b_1}{b_3}, \qquad x = \frac{a_1 + \lambda b_1}{a_3 + \lambda b_3},$$
$$y_1 = \frac{a_2}{a_3}, \qquad y_2 = \frac{b_2}{b_3}, \qquad y = \frac{a_2 + \lambda b_2}{a_3 + \lambda b_3},$$

reduce to

$$(a_3 b_1 - a_1 b_3)(a_3 \mu + \lambda b_3) = 0, \qquad (a_3 b_2 - a_2 b_3)(a_3 \mu + \lambda b_3) = 0.$$

If $a_3 b_1 - a_1 b_3$ and $a_3 b_2 - a_2 b_3$ were both zero, x_1 and y_1 would be equal, respectively, to x_2 and y_2, and the points a and b would not be distinct. Hence $a_3 \mu + \lambda b_3 = 0$, and μ has the desired value.

2. Harmonic Division of Points. Two distinct points Q_1, Q_2 on a finite line L are said to separate harmonically two distinct finite points P_1, P_2 on L if the algebraic ratios μ_1, μ_2 in which they divide the line-segment P_1P_2 are negatives of one another:

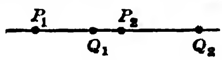


FIG. 1

$$\mu_1 = -\mu_2 \qquad \text{or} \qquad \frac{\overline{Q_1P_1}}{\overline{Q_1P_2}} = -\frac{\overline{Q_2P_1}}{\overline{Q_2P_2}}. \tag{1}$$

If $\mu_2 = 1$, that is, if Q_2 is the point at infinity on L, then $\mu_1 = -1$ and Q_1 is the mid-point of P_1P_2. *Two distinct finite points are separated harmonically by their mid-point and the point at infinity on their line.*

Let P_1, P_2 now have the coordinates $a : (a_1, a_2, a_3)$, $b : (b_1, b_2, b_3)$. Two arbitrary points on their line are $a + \lambda_1 b$, $a + \lambda_2 b$. The ratios μ_1, μ_2 in which they divide P_1P_2 are, by § 1, Th. 2,

$$\mu_1 = -\lambda_1 \frac{b_3}{a_3}, \qquad \mu_2 = -\lambda_2 \frac{b_3}{a_3}.$$

Evidently, $\mu_1 = -\mu_2 (\neq 0)$ if and only if $\lambda_1 = -\lambda_2 (\neq 0)$. Hence:

THEOREM 1. *Two points separate the distinct points a and b harmonically when and only when their coordinates can be written in the forms*

$$a + \lambda b, \qquad a - \lambda b, \qquad \lambda \neq 0,$$

or in the equivalent forms

$$k a + l b, \qquad k a - l b, \qquad k l \neq 0.$$

The definition of harmonic separation assumes that the points P_1 and P_2 are both finite, and no longer makes sense when one or both of these points are ideal. Theorem 1 does, however, still have a meaning in the cases in question. We take the content of the theorem as constituting the definition of harmonic separation in these cases. The theorem is then universally valid.

THEOREM 2. *If Q_1, Q_2 separate P_1, P_2 harmonically, P_1, P_2 separate Q_1, Q_2 harmonically.*

A proof in the case that all four points are finite consists simply in rewriting (1), with due regard for signs, in the form

$$\frac{P_1Q_1}{P_1Q_2} = -\frac{P_2Q_1}{P_2Q_2}.$$

To give a proof which is valid also in case one or all the points are ideal, we take the coordinates of the points in the forms specified by Th. 1:

$$a, \qquad b, \qquad a + \lambda b, \qquad a - \lambda b, \qquad \lambda \neq 0.$$

Setting

$$a' = a + \lambda b, \qquad b' = a - \lambda b,$$

we find that

$$2 a = a' + b', \qquad 2 \lambda b = a' - b'.$$

Hence we can also write, as coordinates of the four points,

$$a' + b', \qquad a' - b', \qquad a', \qquad b'.$$

Consequently, by Th. 1, P_1, P_2 separate Q_1, Q_2 harmonically.

Theorem 2 says that the order of the two pairs of points P_1, P_2 and Q_1, Q_2 is immaterial. So also is the order of the points in each pair; for, it is evident from Th. 1 that, if Q_1 and Q_2 separate P_1, P_2 harmonically, Q_2 and Q_1 separate P_1, P_2 harmonically.

If the points of one pair, say P_1 and P_2, and one of the points of the other pair, say Q_1, are given, the fourth point, Q_2, is uniquely determined. For, if P_1 and P_2 are the points a and b, Q_1 has unique coordinates of the form $a + \lambda b$. Then Q_2 must, by Th. 1, be the point $a - \lambda b$.

Theorem 3. *If P_1, P_2 and Q_1 are distinct collinear points, there is a unique point Q_2 which with Q_1 separates P_1, P_2 harmonically.*

The point Q_2 is known as the *fourth harmonic point* to P_1, P_2 and Q_1, or as the *harmonic conjugate* of Q_1 with respect to P_1 and P_2. The two pairs of points P_1, P_2 and Q_1, Q_2 are said to form a *harmonic set.*

EXERCISES

1. Show that $P_1:(3, 1)$, $P_2:(7, 5)$ and $Q_1:(6, 4)$, $Q_2:(9, 7)$ form a harmonic set, (*a*) by finding the ratios in which Q_1, Q_2 divide P_1P_2; (*b*) by expressing the homogeneous coordinates of Q_1 and Q_2 as linear combinations of those of P_1 and P_2.

2. Prove that $(2, 3, 2)$, $(1, -2, 3)$ and $(8, 5, 12)$, $(4, 13, 0)$ form a harmonic set.

3. Find the harmonic conjugate of the point $(4, 3, 3)$ with respect to the points $(2, 1, 1)$, $(1, 2, 2)$.

4. Find the fourth harmonic point to the points at infinity in the directions of the axes and the point at infinity in the direction of slope unity.

5. Prove that, if P_1, P_2, Q_1, Q_2 are four distinct collinear finite points, Q_1, Q_2 separate P_1, P_2 harmonically if and only if

$$\overline{MQ_1} \cdot \overline{MQ_2} = a^2,$$

where $a = \frac{1}{2} P_1P_2$ and M is the mid-point of P_1P_2.

6. Show that, if Q_1, Q_2 separate P_1, P_2 harmonically, every circle through Q_1, Q_2 cuts the circle described on P_1P_2 as a diameter orthogonally.

3. Division of Two Lines. If L_1 and L_2 are two finite lines and L is a finite line through their point of intersection, the ratio

$$\mu = \frac{D_1}{D_2} \tag{1}$$

of *directed* distances D_1 and D_2 from L_1 and L_2 to an arbitrary point on L is constant. This constant ratio we shall speak of as *the ratio in which the line L divides L_1 and L_2.*

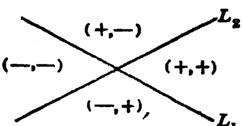


Fig. 2

We do well to enlarge on the first statement. If L_1 and L_2 intersect in a finite point, D_1 and D_2 will both be positive for all points of one of the four regions bounded by L_1 and L_2, both negative for all points of the opposite region, and so forth, as shown in Fig. 2. If the given line L passes through the regions $(+, +)$ and $(-, -)$, the ratio μ defined by (1) is clearly constant and positive. On the other hand, μ is a negative constant, if L passes through the other two regions.

When L_1 and L_2 are parallel, directed distances D_1 and D_2 from L_1 and L_2 to a point P are measured along the same line. We agree to take one and the same direction on this line as the positive direction for measuring *both* D_1 and D_2. Then the signs of D_1 and D_2 for the three regions determined by L_1 and L_2 are, for example, as shown in Fig. 3. In this case, if the given line L, parallel to L_1 and L_2, lies in either of the regions $(+, +)$ or $(-, -)$, the ratio μ is a positive constant, while if L lies in the region $(+, -)$, μ is a negative constant.

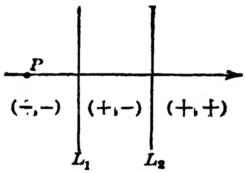


FIG. 3

In the case exemplified by Fig. 3, there is no finite line L corresponding to the ratio $\mu = 1$. It is reasonable, however, to take as the line corresponding to $\mu = 1$ the line at infinity, and we agree to do this.

Inspection shows that there is now always a unique line through a point, finite or ideal, which divides two finite lines through the point in a given ratio.

Remark. The sign of the ratio μ depends on the particular directed distances chosen. Thus, if instead of D_2, we took $-D_2$, μ would change sign. In any particular case, D_1 and D_2 should be chosen first and adhered to throughout. The signs of D_1 and D_2 for each of the regions are then fixed, and hence the position of L for any given value of μ is fixed.*

A Formula for μ. Let L_1 and L_2 have the equations

$$(2)\qquad \alpha \equiv a_1x + a_2y + a_3 = 0, \qquad \beta \equiv b_1x + b_2y + b_3 = 0,$$

and choose, as D_1 and D_2 (Ch. III, § 10, Ex. 1):

$$(3)\qquad D_1 = \frac{\alpha(x, y)}{\sqrt{a_1^2 + a_2^2}}, \qquad D_2 = \frac{\beta(x, y)}{\sqrt{b_1^2 + b_2^2}}.$$

If L is a *finite* line dividing L_1 and L_2 in the ratio μ, the equation of L is, by (1),

$$\frac{\alpha(x, y)}{\sqrt{a_1^2 + a_2^2}} = \mu \frac{\beta(x, y)}{\sqrt{b_1^2 + b_2^2}},$$

* In this connection, we note that if, in the case in which L_1 and L_2 are parallel, we had agreed to take the positive directions for measuring D_1 and D_2 opposite to one another, the sign of one of the distances in Fig. 3, say that of D_2, would be reversed and the pairs of signs in the three regions would read, from left to right, $(-, +)$, $(+, +)$, $(+, -)$. Then, to the line at infinity we should make correspond, not $\mu = 1$, but $\mu = -1$.

or

$$(4) \qquad \alpha + \lambda\beta = 0 \qquad \text{where} \qquad \lambda = -\mu\sqrt{\frac{a_1^2 + a_2^2}{b_1^2 + b_2^2}}.$$

THEOREM 1. *The ratio μ in which the line $\alpha + \lambda\beta = 0$ divides the finite lines $\alpha = 0$, $\beta = 0$ is*

$$(5) \qquad \mu = -\lambda\sqrt{\frac{b_1^2 + b_2^2}{a_1^2 + a_2^2}},$$

provided that it is understood that D_1 and D_2 are chosen as in (3).

If $\alpha + \lambda\beta = 0$ is a finite line, the theorem follows directly from (4).*

4. Harmonic Division of Lines. Two distinct lines M_1, M_2 through a point P, finite or ideal, are said to separate harmonically two finite distinct lines L_1, L_2 through P if the algebraic ratios μ_1, μ_2 in which they divide L_1, L_2 are negatives of one another:

$$(1) \qquad \mu_1 = -\mu_2.$$

If P is a finite point and $\mu_2 = 1$, $\mu_1 = -1$, then M_1 and M_2 are the bisectors of the angles between L_1 and L_2. *The bisectors of the angles between two lines intersecting in a finite point separate the lines harmonically.*

If L_1 and L_2 are the lines $\alpha = 0$, $\beta = 0$ of § 3, but with their equations now written in homogeneous coordinates:

$$\alpha \equiv a_1x_1 + a_2x_2 + a_3x_3 = 0, \qquad \beta \equiv b_1x_1 + b_2x_2 + b_3x_3 = 0,$$

two arbitrary lines through P are $\alpha + \lambda_1\beta = 0$, $\alpha + \lambda_2\beta = 0$. The ratios μ_1, μ_2 in which these lines divide L_1, L_2 are negatives of one another, by § 3, Th. 1, if and only if $\lambda_1 = -\lambda_2$. Hence:

* To verify the theorem when $\alpha = 0$, $\beta = 0$ are parallel and $\alpha + \lambda\beta = 0$ is the line at infinity—to obtain the line at infinity as a linear combination of $\alpha = 0$, $\beta = 0$, we think of the equations (2) as written in homogeneous coordinates—we must show that (5) reduces to $\mu = +1$ or $\mu = -1$, according as the positive directions for measuring the distances (3) are the same or opposite; see preceding footnote.

Inasmuch as $\alpha = 0$ and $\beta = 0$ are parallel, a constant k, $\neq 0$, exists so that $a_1 = k\,b_1$, $a_2 = k\,b_2$. If, then, $\alpha + \lambda\beta = 0$ is to be the line at infinity, λ must have the value $-k$. For $a_1 = k\,b_1$, $a_2 = k\,b_2$, and $\lambda = -k$, (5) reduces to $\mu = k/\sqrt{k^2}$. Consequently, $\mu = 1$ or $\mu = -1$ according as $k > 0$ or $k < 0$.

It remains to show that, according as $k > 0$ or $k < 0$, the positive directions for measuring the distances (3) are the same or opposite. From the formulas (3) we readily find that, if $k > 0$, $D_1 - D_2 \equiv$ const., whereas if $k < 0$, $D_1 + D_2 \equiv$ const. Hence, the positive directions for measuring the two distances are the same in the first case and opposite in the second.

THEOREM 1. *Two lines separate the distinct lines $\alpha = 0$, $\beta = 0$ harmonically if and only if their equations can be written in the forms*

$$\alpha + \lambda\beta = 0, \qquad \alpha - \lambda\beta = 0, \qquad \lambda \neq 0,$$

or in the equivalent forms

$$k\alpha + l\beta = 0, \qquad k\alpha - l\beta = 0, \qquad kl \neq 0.$$

The theorem has been proved in case L_1, L_2 are both finite lines. We take the content of it as the definition of harmonic division in case one of these lines is the line at infinity.

The following theorems are the analogs of Theorems 2 and 3 of § 2, and are similarly proved.

THEOREM 2. *If M_1, M_2 separate L_1, L_2 harmonically, L_1, L_2 separate M_1, M_2 harmonically.*

THEOREM 3. *If L_1, L_2 and M_1 are distinct concurrent lines, there is a unique line M_2 which with M_1 separates L_1, L_2 harmonically.*

The line M_2 is called the fourth harmonic line to L_1, L_2 and M_1, or the harmonic conjugate of M_1 with respect to L_1, L_2. The two pairs of lines are said to form a harmonic set.*

EXERCISES

1. Show that the pairs of lines,

$$\begin{aligned} 2x_1 - 3x_2 + 4x_3 &= 0, \\ x_1 + x_2 + x_3 &= 0, \end{aligned} \qquad \text{and} \qquad \begin{aligned} 7x_1 - 3x_2 + 11x_3 &= 0, \\ x_1 - 9x_2 + 5x_3 &= 0, \end{aligned}$$

form a harmonic set.

2. Show that the pairs of lines

$$x = 0, \quad y = 0 \qquad \text{and} \qquad x + 2y = 0, \quad x - 2y = 0$$

form a harmonic set.

3. Find the harmonic conjugate of $4x_1 - 3x_2 + 4x_3 = 0$ with respect to $x_1 - 2x_2 + x_3 = 0$, $2x_1 + x_2 + 2x_3 = 0$.

4. Prove that two perpendicular lines are separated harmonically by any two lines which pass through their common point and are equally inclined to each of them.

5. Harmonic Division a Projective Property. We first establish the following theorems.

* Harmonic division and its generalization, cross ratio (Ch. VI), were known to the Greeks. The introduction of directed line-segments and the use of negative, as well as positive, quantities in geometry are, however, modern ideas, first carried through in a consequential manner by Moebius (1790–1868) in his principal work, *Der barycentrische Calcül*, 1829.

Theorem 1. *If four lines, when properly paired, form a harmonic set, the four points in which they are cut by a transversal, when correspondingly paired, form a harmonic set.*

Theorem 2. *If four points, properly paired, form a harmonic set, the four lines joining them to a fifth point, when correspondingly paired, form a harmonic set.*

The two theorems can be proved simultaneously. Let P_1, P_2, Q_1, Q_2 be four collinear points with the coordinates

$$(1) \qquad a, \qquad b, \qquad a + \lambda b, \qquad a + \lambda' b.$$

Let the lines joining these points to an arbitrary point $r : (r_1, r_2, r_3)$ external to their line be L_1, L_2, M_1, M_2. The equations of L_1 and L_2 are $|x\,a\,r| = 0$, $|x\,b\,r| = 0$. The equation of M_1 is

$$|x\; \overline{a + \lambda b}\; r| = 0$$

or

$$|\,x\,a\,r| + \lambda |x\,b\,r| = 0.$$

Similarly, for that of M_2. Thus L_1, L_2, M_1, M_2 have the equations

$$(2) \qquad \begin{aligned} |x\,a\,r| &= 0, \qquad & |x\,a\,r| + \lambda\,|x\,b\,r| &= 0, \\ |x\,b\,r| &= 0, \qquad & |x\,a\,r| + \lambda' |x\,b\,r| &= 0. \end{aligned}$$

The condition that P_1, P_2 and Q_1, Q_2 form a harmonic set is that $\lambda' = -\lambda$. But this is precisely the condition that L_1, L_2 and M_1, M_2 form a harmonic set. Hence both theorems are proved.

Theorem 3. *Harmonic division is a projective property.*

In the case of a projection of a line on a line (Fig. 4), we have to show that, if the pairs of points P_1, P_2 and Q_1, Q_2 form a harmonic set, their projections P_1', P_2' and Q_1', Q_2' form a harmonic set. Since P_1, P_2 and Q_1, Q_2 form a harmonic set, the lines CP_1, CP_2 and CQ_1, CQ_2 form a harmonic set, by Th. 2. But then, by Th. 1, the points P_1', P_2' and Q_1', Q_2' form a harmonic set.

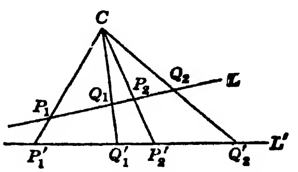


Fig. 4

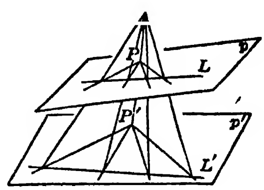


Fig. 5

In the case of a projection of a plane p on a plane p', we have merely to show that a harmonic set of four lines projects into a harmonic

set of four lines. Given, in Fig. 5, that the four lines through P in the plane p form a harmonic set, to prove that their projections on p'—the four lines through P'— form a harmonic set. Since the four lines through P form a harmonic set, the four points in which they are cut by a transversal L form a harmonic set. Consequently, the four points in which the projection, L', of L cuts the four lines through P' form a harmonic set. But then so do these lines themselves.

The fact that harmonic division is a projective property could hardly have been foreseen from its definition. For this definition was based on the ratios of distances, and distance is, in itself, a metric property.*

All previous projective properties were capable of dualization. Accordingly, it is natural to agree that *a harmonic set of four points and a harmonic set of four lines shall constitute dual figures.* Theorems 1 and 2 are, then, dual theorems.

EXERCISES

1. Three distinct lines, L_1, L_2 and M_1, through a finite point O, are given. A line is drawn parallel to M_1 meeting L_1 and L_2 in P_1 and P_2. The mid-point of P_1P_2 is joined to O by the line M_2. Prove that M_2 is the fourth harmonic line to L_1, L_2 and M_1.

2. Three distinct points P_1, P_2 and Q_1 are given, on a line L. Parallel lines, L_1 and L_2, are drawn through P_1 and P_2, and a line is passed through Q_1 meeting L_1 and L_2 in A_1 and A_2. The point B_2 on L_2 is then marked so that P_2 is the mid-point of A_2B_2, and A_1B_2 is drawn meeting L in Q_2. Show that Q_2 is the harmonic conjugate of Q_1 with respect to P_1, P_2.

6. Harmonic Properties of the Complete Quadrilateral and Complete Quadrangle. Connected with the complete quadrilateral and the complete quadrangle are numerous harmonic sets of points and lines, which are of considerable importance in projective geometry.

Complete Quadrilateral. Two vertices of a complete quadrilateral whose join is not a side of the quadrilateral are called *opposite vertices.*

* The reader may have noticed that the development of the theory of harmonic division lacked unity and elegance in that first finite elements were considered and the results then extended to hold for ideal elements. The reason for this is now clear. We were really developing a projective property, but deemed it wise to base the development on the metric concept of distance. It is only in the finite plane that distance is defined, whereas projective properties must be defined throughout the extended plane. Consequently, the concept of harmonic division, developed first for finite elements, had to be extended to hold for all elements.

There are three pairs of opposite vertices: P_1, P_2; Q_1, Q_2; R_1, R_2 (Fig. 6). The three lines P_1P_2, Q_1Q_2, R_1R_2, or p, q, r, determined by the pairs of opposite vertices are known as *diagonals.* Finally, the triangle PQR formed by the three diagonals is called the *diagonal triangle.*

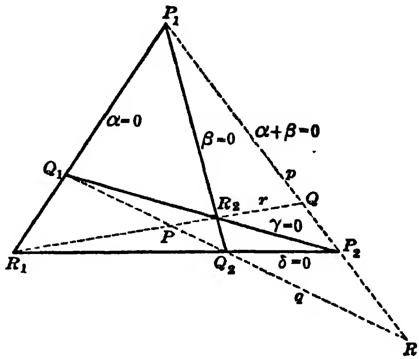


Fig. 6

According to Ch. III, § 5, Ex. 5, equations $\alpha = 0, \beta = 0$, $\gamma = 0$, $\delta = 0$ of the sides of the complete quadrilateral can be chosen so that

(1) $\alpha + \beta + \gamma + \delta \equiv 0.$

Making use of this identity, we obtain two equations for each of the diagonals, as follows:

$$(2) \qquad \begin{array}{llcl} p: & \alpha + \beta = 0 & \text{or} & \gamma + \delta = 0, \\ q: & \alpha + \gamma = 0 & \text{or} & \delta + \beta = 0, \\ r: & \alpha + \delta = 0 & \text{or} & \beta + \gamma = 0. \end{array}$$

For example, since $\alpha + \beta \equiv -(\gamma + \delta)$, the equations $\alpha + \beta = 0$ and $\gamma + \delta = 0$ represent the same line and this line is evidently the diagonal p.

Consider now the two sides of the quadrilateral which intersect in P_1 and the diagonal p which goes through P_1. The harmonic conjugate of the diagonal, $\alpha + \beta = 0$, with respect to the sides, $\alpha = 0$, $\beta = 0$, has the equation $\alpha - \beta = 0$. What line is this? The figure suggests P_1P, that is, the line joining P_1 to the point of intersection of the other two diagonals, q and r. If this is correct, we should be able to show that $\alpha - \beta = 0$ is a linear combination of one of the equations of q and one of the equations of r. This it is, for

$$\alpha - \beta \equiv (\alpha + \gamma) - (\beta + \gamma).$$

Thus, we have established the following harmonic property of the complete quadrilateral.

Theorem 1a. *Two sides of a complete quadrilateral are separated harmonically by the diagonal through their point of intersection and the line joining their point of intersection with the point of intersection of the other two diagonals.*

Let the reader now draw the line P_1P.

Since the pairs of lines at P_1 form a harmonic set, the pairs of points in which they are cut by the diagonal r, namely R_1, R_2 and P, Q, form a harmonic set. Hence, we have

THEOREM 2 *a*. *Two opposite vertices of a complete quadrilateral are separated harmonically by the points in which their diagonal is met by the other two diagonals.*

Since R_1, R_2 and P, Q form a harmonic set, the pairs of lines joining them to the point R form a harmonic set. Two of these lines, RQ and RP, are the diagonals p and q; the other two are the lines joining R to the vertices R_1, R_2 which lie on the third diagonal, r.

THEOREM 3 *a*. *Two diagonals of a complete quadrilateral are separated harmonically by the lines joining their point of intersection with the vertices on the third diagonal.*

Complete Quadrangle. The results here are precisely the duals of those just established.

THEOREM 1 *b*. *Two vertices of a complete quadrangle are separated harmonically by the diagonal point on their side and the point of intersection of their side with the line joining the other two diagonal points.*

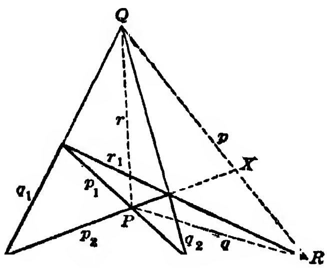


FIG. 7

To prove the theorem, we do not have to duplicate, in the dual, the proof of Theorem 1 *a*. We have to show, for example, that the two vertices lying on the side p_2 (Fig. 7) are separated harmonically by the diagonal point P and the point X in which p_2 meets the line p. But this follows immediately from the fact that, in the complete quadrilateral consisting of the lines q_1, q_2, r_1, r_2, the sides q_1, q_2 are separated harmonically by p and r.

The statement and proof of Theorem 2 *b*, the dual of Theorem 2 *a*, we leave to the reader.

THEOREM 3 *b*. *Two diagonal points of a complete quadrangle are separated harmonically by the points in which their line is intersected by the sides passing through the third diagonal point.*

EXERCISES

1. State and prove Theorem 2 *b*. **2.** Prove Theorem 3 *b*.

3. Prove Theorem 1 *b* analytically by dualizing the proof of Theorem 1 *a*.

4. Let A, B, C be three collinear vertices of a complete quadrilateral, and let M be the point which with C separates A and B harmonically. Prove that the line joining M with the vertex opposite to C goes through the point of intersection of the two diagonals which pass through A and B respectively. State the dual theorem.

7. Construction of the Fourth Harmonic Element to Three Given Elements. Projective Criterion for Harmonic Separation. When three collinear points P_1, P_2 and Q_1, are given, the point Q_2 which, with Q_1, separates P_1, P_2 harmonically may be constructed in the following manner.

On a line through Q_1 choose two points, A and C (Fig. 8), and draw the lines joining A and C to P_1 and P_2. Denote the new intersections of these lines by B and D. Then the line BD meets the line of the given points in the required point, Q_2. For, P_1 and P_2 are two diagonal points of the complete quadrangle $ABCD$, and Q_1 and Q_2 are the points in which their line is intersected by the sides of the quadrangle through the third diagonal point. Consequently, by § 6, Th. 3 *b*, Q_1, Q_2 separate P_1, P_2 harmonically.

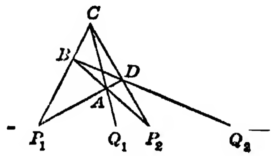


FIG. 8

The construction tells us that, if P_1, P_2 and Q_1, Q_2 form a harmonic set, there exists a complete quadrangle $ABCD$ with two sides passing through each of the points P_1, P_2 and one side through each of the points Q_1, Q_2. Theorem 3 *b* of the preceding paragraph tells us, conversely, that two pairs of points which have this relationship to a complete quadrangle form a harmonic set.

THEOREM 1 *a*. *Two pairs of collinear points form a harmonic set if and only if there exists a complete quadrangle with two sides through each of the points of the one pair and one side through each of the points of the other pair.*

We have in this theorem a criterion for a harmonic set of four points which is purely projective. If we had seen fit to develop the theory of harmonic separation without recourse to the metric property of distance, we might well have started with the content of this theorem as the definition of harmonic division.

EXERCISES

1. Give a construction for the fourth harmonic line to three given concurrent lines.

2. State and prove the dual of Theorem 1 *a*.

3. Prove: Two pairs of points form a harmonic set if and only if there exists a complete quadrilateral which has the points of the one pair as two opposite vertices, and the points of the other pair as the intersections of the diagonal of these vertices with the other two diagonals. State the dual.

8. A Projective Generalization of a Metric Theorem. The reader is familiar with the theorem to the effect that the medians of a triangle are concurrent. We propose to generalize this theorem, that is, to obtain a more general theorem of which it is but a special case.

The mid-point of a side L of a triangle may be thought of as the harmonic conjugate, with respect to the vertices on L, of the point at infinity on L. The theorem of the medians may, then, be constructed as follows. Start with the points at infinity on the sides of the triangle and mark their harmonic conjugates with respect to the vertices—the mid-points of the sides; then the lines joining these points to the opposite vertices—the medians—are concurrent.

The points at infinity on the sides of the triangle are three collinear points, one on each side of the triangle. If we take, instead of them, any three distinct collinear points, one on each side of the triangle, the theorem still remains true.

THEOREM 1. *If three distinct points, one on each side of a triangle, are collinear, the three lines which join their harmonic conjugates, with respect to the vertices of the triangle, to the opposite vertices, are concurrent.*

Let $P_1P_2P_3$ be the given triangle, Q_1, Q_2, Q_3 the three collinear points, one on each side, and R_1, R_2, R_3 their harmonic conjugates with respect to the vertices, as shown in Fig. 9.

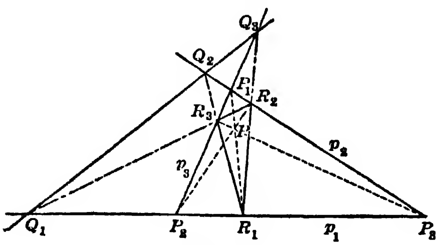


FIG. 9

The figure, when accurately drawn, suggests that the sides of the triangle $R_1R_2R_3$ intersect the corresponding sides of the given triangle in the points Q_1, Q_2, Q_3. If this is the case, the two triangles are in the relationship of Desargues, the lines P_1R_1, P_2R_2, P_3R_3 joining corresponding vertices are concurrent, and the theorem is proved.

The figure actually states the facts. To show, for example, that Q_1, R_3, R_2 are collinear, we have merely to project the line p_3, from Q_1, on the line p_2. The points P_1, P_2, and Q_3 project into the points P_1, P_3, and Q_2. Since R_3 is the fourth harmonic point to P_1, P_2 and Q_3, it must project into the fourth harmonic point to P_1, P_3 and Q_2, that is, R_2. Consequently Q_1, R_3, and R_2 are collinear.

It is to be noted that, whereas the theorem of the medians is a metric theorem, the generalization of it which we have established is a projective theorem.

Converse of Theorem 1. *If the lines joining three distinct points, one on each side of a triangle, to the opposite vertices are concurrent, the harmonic conjugates of the three points, with respect to the vertices of the triangle, are collinear.*

Since, by hypothesis, P_1R_1, P_2R_2, P_3R_3 have a common point P, the points Q_1, Q_2, Q_3 will be collinear if it can be shown that they are the intersections of the pairs of corresponding sides of the triangles $P_1P_2P_3$ and $R_1R_2R_3$. To prove, say, that R_2R_3 and P_2P_3 intersect in Q_1, consider the complete quadrangle $P_1R_2PR_3$. Two sides pass through each of the points P_2, P_3 and one side passes through the point R_1. Consequently, by § 7, the sixth side, R_2R_3, must pass through the harmonic conjugate of R_1 with respect to P_2, P_3, that is, through Q_1. Hence R_2R_3 intersects P_2P_3 in Q_1, as desired.

EXERCISES

1. Dual of Theorem 1. *If three distinct lines, one through each vertex of a triangle, are concurrent, the three points in which their harmonic conjugates, with respect to the sides of the triangle, meet the opposite sides are collinear.* Prove this theorem.

2. What metric theorem results when the bisectors of the interior angles of the triangle are taken as the three given lines?

3. Show that the dual of Theorem 1 is essentially the same as the converse of Theorem 1.

4. Give an analytic proof of Theorem 1. Take, as the coordinates of P_1, P_2, P_3, $a : (a_1, a_2, a_3)$, b, c. The coordinates of Q_1, Q_2, Q_3 will then be $Bb - Cc$, $Cc - Aa$, $Aa - Bb$, where A, B, C are constants (Ch. III, § 7, Ex. 4). Hence R_1, R_2, R_3 will have the coordinates $Bb + Cc$, $Cc + Aa$, $Aa + Bb$. (Why?)

Finally, find by inspection a point P whose coordinates are a linear combination of a and $Bb + Cc$, also of b and $Cc + Aa$, and finally of c and $Aa + Bb$; or write the equations of the three lines P_1R_1, P_2R_2, P_3R_3 and show them linearly dependent.

5. Give an analytic proof of the dual of Theorem 1.

6. Prove that if three distinct points, one on each side of the triangle with vertices a, b, c, are given, the lines joining the points to the opposite vertices are concurrent if and only if the coordinates of the points can be written in the forms $Bb + Cc$, $Cc + Aa$, $Aa + Bb$; see Ex. 4.

7. State the dual of the preceding theorem.

8. If three distinct lines, one through each vertex of a triangle, are concurrent, the harmonic conjugates, with respect to the sides of the triangle, of two of the lines intersect on the third. Prove this theorem. What metric theorem can be obtained from it as a special case?

9. Show that if three lines, one through each vertex of a triangle, are concurrent, their harmonic conjugates with respect to the sides of the triangle form a second triangle which is in the relationship of Desargues with the first.

10. *Construction of a complete quadrilateral with a given diagonal triangle.* Show that, if Q_1, Q_2, Q_3 are three distinct collinear points, one on each side of a triangle, and R_1, R_2, R_3 are their harmonic conjugates with respect to the vertices, then Q_1, R_1, Q_2, R_2, and Q_3, R_3 are the pairs of opposite vertices of a complete quadrilateral whose diagonal triangle is the given triangle.

11. Describe and justify the construction of a complete quadrangle with a given diagonal triangle.

12. Prove that if three distinct points, one on each side of the diagonal triangle of a complete quadrilateral, are collinear, their harmonic conjugates with respect to the vertices of the quadrilateral are also collinear.

Suggestion for an analytic proof. Begin by assigning coordinates to the vertices of the diagonal triangle, and expressing, in terms of these coordinates, the coordinates of the vertices of the quadrilateral; see Ex. 10.

13. Obtain a theorem concerning the mid-points of the diagonals of a complete quadrilateral from the theorem of Ex. 12.

CHAPTER V

LINE COORDINATES

1. Point Geometry and Line Geometry. In the geometry which we have thus far studied, the point has always been the fundamental element. It was to the point that we gave coordinates and it was as loci of points that we studied curves.

We have learned, however, that the straight line, at least in so far as the principle of duality holds sway, is as important as the point. There must exist, then, in contrast to the familiar geometry of points, an equally significant geometry of lines: a geometry in which the line is the fundamental element.

In this geometry the point must be thought of as defined by lines. In particular, since in point geometry a line is a range of points, in line geometry a point should appear as a pencil of lines. Later we shall learn to think of an arbitrary curve, from the point of view of line geometry, as defined, not by its points, but by its tangent lines.

Our initial problem in the development of line geometry is to assign coordinates to the lines of the plane.

2. Homogeneous Line Coordinates. We recall the theorem: Every linear homogeneous equation in x_1, x_2, x_3,

$$(1)\qquad a_1x_1 + a_2x_2 + a_3x_3 = 0,$$

where a_1, a_2, a_3 are not all zero, represents a straight line, and conversely.

The equations

$$(2)\quad 2\,x_1 - 3\,x_2 + 4\,x_3 = 0, \qquad 2\,r\,x_1 - 3\,r\,x_2 + 4\,r\,x_3 = 0, \qquad r \neq 0,$$

represent the same line. Coordinates of this line must serve to distinguish it from all other lines. The sets of coefficients in its equations surely do this. What simpler choice could we make, then, for the coordinates than these sets of coefficients?

Definition. *The coefficients in an equation of a line shall constitute a set of homogeneous coordinates of the line.*

Thus, $(2, -3, 4)$, $(4, -6, 8)$, and $(2\,r, -3\,r, 4\,r)$ are sets of homogeneous coordinates of the line (2). The triples (a_1, a_2, a_3) and $(r\,a_1, r\,a_2, r\,a_3)$ serve the same purpose for the arbitrary line (1).

It is evident that every line has infinitely many sets of homogeneous coordinates, each two sets being proportional. Conversely, any three numbers, definitely ordered and not all zero, are homogeneous coordinates of a unique line. Thus, (2, 0, 3) are coordinates of the line $2x_1 + 3x_3 = 0$.

We shall use (u_1, u_2, u_3) to denote homogeneous coordinates of an arbitrary line and we shall call this line the line u.

THEOREM 1. *The point $x : (x_1, x_2, x_3)$ lies on the line $u : (u_1, u_2, u_3)$ if and only if*

$$u_1x_1 + u_2x_2 + u_3x_3 = 0. \tag{3}$$

For, an equation of the line u is

$$u_1X_1 + u_2X_2 + u_3X_3 = 0,$$

where (X_1, X_2, X_3) are "running coordinates," and the point x lies on this line if and only if (3) holds.

DEFINITION. *An equation in line coordinates of a given point is an equation in u_1, u_2, u_3 which is satisfied by the coordinates of those and only those lines which pass through the point.*

Geometrically, the definition amounts to saying that, in line geometry, a point is to be thought of as a pencil of lines.

According to Th. 1, the line u goes through the point (2, 1, 3) if and only if

$$2u_1 + u_2 + 3u_3 = 0.$$

Hence, this is an equation in line coordinates of the point (2, 1, 3).

THEOREM 2. *An equation, in line coordinates, of the point $a : (a_1, a_2, a_3)$ is*

$$a_1u_1 + a_2u_2 + a_3u_3 = 0.$$

Conversely, every linear homogeneous equation in u_1, u_2, u_3, whose coefficients are not all zero, represents a point.

The proof of the theorem is left to the reader.

Analytic Duality between Point and Line. The fundamental aspects of point geometry and line geometry are now before us. In point geometry a point has coordinates and a line an equation, whereas in line geometry it is the line which has coordinates and the point an equation in these coordinates.

The two geometries are related by means of the all important condition (3) that the point x lie on the line u. The relationship is reciprocal, inasmuch as (3) is symmetric in the x's and u's. Not only

are the coordinates of a line the coefficients in its equation in point coordinates; the coordinates of a point are also the coefficients in its equation in line coordinates. Thus:

	Coordinates	*Equation*
Point	$a : (a_1, a_2, a_3)$	$a_1u_1 + a_2u_2 + a_3u_3 = 0,$
Line	$a : (a_1, a_2, a_3)$	$a_1x_1 + a_2x_2 + a_3x_3 = 0.$

The reciprocal relationship between point and line is an analytic reproduction of the geometric reciprocity between point and line. In other words, we have laid the foundation for an analytic duality corresponding to the geometric duality with which we have long been familiar.*

EXERCISES

1. What are the coordinates of the y-axis? The line at infinity? The line through the origin of slope 2?

2. Identify in each case the line which has the given coordinates:

$$(a)\quad (1, 1, -1); \qquad (b)\quad (1, -1, 0); \qquad (c)\quad (0, 1, 0).$$

3. What does each of the following equations represent?

$$2u_1 - 3u_2 + u_3 = 0; \qquad u_2 - u_3 = 0; \qquad u_1 = 0.$$

4. What is the equation in line coordinates of the origin? Of the point at infinity in the direction of slope 1/2?

3. A Notation. The analytic geometry of the extended plane, as developed by means of homogeneous coordinates of points and lines, deals with ordered triples of numbers: (a_1, a_2, a_3), (x_1, x_2, x_3), (u_1, u_2, u_3). Combinations of these triples which have proved important are

$$a_1x_1 + a_2x_2 + a_3x_3, \qquad u_1x_1 + u_2x_2 + u_3x_3.$$

Before proceeding further, it is wise to adopt a suitable notation for expressions of this type.

If $a : (a_1, a_2, a_3)$, $b : (b_1, b_2, b_3)$ are two arbitrary ordered triples of numbers, we write for the expression

$$a_1b_1 + a_2b_2 + a_3b_3$$

the symbol $(a|b)$, to be read "a into b":

$$(a|b) \equiv a_1b_1 + a_2b_2 + a_3b_3.$$

* It is not often that an idea can be ascribed to a single man with the assurance that the concept of the line as fundamental element can be ascribed to Pluecker. In 1829 he introduced line coordinates—just as they are here defined—and developed the analytic duality between point and line.

The equation of the straight line $a : (a_1, a_2, a_3)$ now becomes $(a|x) = 0$, and the condition that the point x lie on the line u becomes $(u|x) = 0$.

EXERCISE. Establish the following laws of operation for the symbol $(a|b)$:

$$(a|b) = (b|a), \qquad (\overline{a + b}|c) = (a|c) + (b|c),$$

$$(ka|b) = k(a|b), \qquad (\overline{ka + lb}|c) = k(a|c) + l(b|c),$$

where k and l are simple numbers.

4. Analytic Duality between Point and Line. We can best exhibit the completeness of the analytic duality between point and line by putting the results in parallel columns.

POINTS		LINES	
Coordinates	*Equations*	*Coordinates*	*Equations*
(a_1, a_2, a_3)	$(a\,|\,u) = 0$	(a_1, a_2, a_3)	$(a\,|\,x) = 0$
(b_1, b_2, b_3)	$(b\,|\,u) = 0$	(b_1, b_2, b_3)	$(b\,|\,x) = 0$
(c_1, c_2, c_3)	$(c\,|\,u) = 0$	(c_1, c_2, c_3)	$(c\,|\,x) = 0.$

We have noted that in testing a number of lines for linear dependence it is immaterial whether we apply the test to the equations of the lines or to the sets of coefficients in the equations, now the coordinates of the lines. Similarly in the case of points, whether we use the coordinates or the equations is immaterial.

THEOREM 1 *a*. *Two points are identical if and only if their coordinates or their equations are linearly dependent.*

THEOREM 2 *a*. *Three points are collinear if and only if their coordinates or their equations are linearly dependent.*

THEOREM 1 *b*. *Two lines are identical if and only if their coordinates or their equations are linearly dependent.*

THEOREM 2 *b*. *Three lines are concurrent if and only if their coordinates or their equations are linearly dependent.*

We consider next the point of intersection of two distinct lines and the line joining two distinct points.

THEOREM 3 *a*. *An equation of the line joining the points a and b is*

$$|x\,a\,b| = 0.$$

Coordinates of this line are

$$|a_2\,b_3|, \quad |a_3\,b_1|, \quad |a_1\,b_2|.$$

THEOREM 3 *b*. *An equation of the point of intersection of the lines a and b is*

$$|u\,a\,b| = 0.$$

Coordinates of this point are

$$|a_2\,b_3|, \quad |a_3\,b_1|, \quad |a_1\,b_2|.$$

The first part of Theorem 3 a is familiar. The second part follows from it, for the coefficients in the equation of a line are coordinates of the line and the coefficients of x_1, x_2, x_3 in $|x\, a\, b| = 0$ are precisely the two-rowed determinants $|a_2\, b_3|$, $|a_3\, b_1|$, $|a_1\, b_2|$.

It is worth while to give a second proof. According to the definition of the equation of a point, a simultaneous solution, other than 0, 0, 0, of the equations of the two points a and b,

$$a_1u_1 + a_2u_2 + a_3u_3 = 0, \qquad b_1u_1 + b_2u_2 + b_3u_3 = 0,$$

is a set of coordinates of the line joining the two points. Hence, $|a_2\, b_3|$, $|a_3\, b_1|$, $|a_1\, b_2|$ are coordinates of the line.

THEOREM 4 a. *The general point of the range determined by the distinct points a and b is $k\,a + l\,b$; or, the equation of an arbitrary point of the range determined by the points $(a|u) = 0$, $(b|u) = 0$ is*

$$k\,(a|u) + l\,(b|u) = 0.$$

THEOREM 4 b. *The general line of the pencil determined by the distinct lines a and b is $k\,a + l\,b$; or, the equation of an arbitrary line of the pencil determined by the lines $(a|x) = 0$, $(b|x) = 0$ is*

$$k\,(a|x) + l\,(b|x) = 0.$$

The first part of Theorem 4 a and the second part of Theorem 4 b are the previous forms of these statements. The other two parts follow directly from these two. For example, the equation of the point $k\,a + l\,b$ is $(\overline{k\,a + l\,b}|u) = 0$, and this reduces to $k(a|u) + l(b|u) = 0$.

Before the introduction of line coordinates we had at our disposal only coordinates for points and equations for lines. We now have coordinates and equations for both. Whether we use coordinates or equations is in general immaterial. We might, if we desired, get along with coordinates alone. The following example illustrates this point of view.

Example. Find the coordinates of the line which joins the point $(1, 2, -1)$ with the point of intersection of the lines $(2, 1, 3)$, $(1, -1, 0)$.

An arbitrary line through the point of intersection of the two given lines has, by Th. 4 b, the coordinates $(2\,k + l,\ k - l,\ 3\,k)$. This line goes through the point $(1, 2, -1)$ if

$$2\,k + l + 2(k - l) - 3\,k = 0,$$

that is, if $k - l = 0$, or if $k = l = 1$. Hence the required line is $(3, 0, 3)$, or $(1, 0, 1)$.

EXERCISES

1. Find the coordinates of the line which joins the points

$$3u_1 + 4u_2 - 11u_3 = 0, \qquad 5u_1 - 3u_2 + u_3 = 0.$$

2. Find the coordinates of the point in which the line $(1, -1, 2)$ is met by the line joining the points $(3, 4, -1)$, $(5, -3, 1)$.

3. Find the coordinates of the line which goes through the point of intersection of the lines $(1, 1, 1)$, $(2, 1, 3)$ and through the point $2u_1+3u_2+u_3=0$.

4. Prove one half of the triangle theorem of Desargues, using only coordinates of lines.

5. Show that coordinates of the point in which the line joining the points a and b is met by the line c are $(b|c)\,a - (a|c)\,b$. State the dual.

6. Show that coordinates of the point in which the line determined by the points a and b intersects the line determined by the points c and d are

$$|b\,c\,d|a - |a\,c\,d|b \qquad \text{or} \qquad |a\,b\,d|c - |a\,b\,c|d.$$

5. Nonhomogeneous Line Coordinates. The nonhomogeneous coordinates (u, v) of the line (u_1, u_2, u_3) shall be defined as the ratios

$$u = \frac{u_1}{u_3}, \qquad v = \frac{u_2}{u_3},$$

provided $u_3 \neq 0$. Since $u_3 = 0$, or $0\,u_1 + 0\,u_2 + u_3 = 0$, is the equation of the origin, the lines for which $u_3 = 0$ are the lines through the origin. These lines have no nonhomogeneous coordinates.*

If, in the relation,

$$u_1x_1 + u_2x_2 + u_3x_3 = 0, \qquad u_3x_3 \neq 0,$$

we divide by u_3x_3 and set $u = u_1/u_3$, $v = u_2/u_3$, $x = x_1/x_3$, $y = x_2/x_3$, we have

$$ux + vy + 1 = 0.$$

THEOREM 1. *A necessary and sufficient condition that the point (x, y) lie on the line (u, v) is that*

$$ux + vy + 1 = 0. \tag{1}$$

From this condition follow the theorems:

THEOREM 2. *The equation, in nonhomogeneous line coordinates, of the point (x_0, y_0), not the origin, is*

$$x_0u + y_0v + 1 = 0. \tag{2}$$

* Compare the facts for point coordinates. The nonhomogeneous coordinates (x, y) of the point (x_1, x_2, x_3) are $x = x_1/x_3$, $y = x_2/x_3$, provided $x_3 \neq 0$. The points for which $x_3 = 0$, that is, the points on the line at infinity, have no nonhomogeneous coordinates.

THEOREM 3. *The equation, in nonhomogeneous point coordinates, of the line* (u_0, v_0), *not the line at infinity, is*

$$u_0 x + v_0 y + 1 = 0. \tag{3}$$

If $u_0 v_0 \neq 0$, the line (3) has intercepts on the axes, namely,

$$a = -\frac{1}{u_0}, \qquad b = -\frac{1}{v_0}.$$

Then

$$u_0 = -\frac{1}{a}, \qquad v_0 = -\frac{1}{b}.$$

Hence, the *nonhomogeneous coordinates of the line are the negative reciprocals of its intercepts on the axes.*

EXERCISES

1. Show that the lines (u_1, v_1) and (u_2, v_2) are parallel if and only if $u_1v_2 - u_2v_1 = 0$, and perpendicular if and only if $u_1u_2 + v_1v_2 = 0$.

2. Show that, if the equations of the two distinct points $P_1 : (x_1, y_1)$, $P_2 : (x_2, y_2)$ are written in the forms

$$\alpha \equiv x_1u_1 + y_1u_2 + u_3 = 0, \qquad \beta \equiv x_2u_1 + y_2u_2 + u_3 = 0,$$

the equation $\alpha - \mu\beta = 0$ represents the point which divides the line-segment P_1P_2 in the ratio μ.

3. Prove that the points in which the bisectors of the exterior angles of a triangle meet the opposite sides are collinear.

4. Show that, if $\alpha = 0, \beta = 0$ are as given in Ex. 2, the points $k\alpha - l\beta = 0$ and $l\alpha - k\beta = 0$ are equally distant from the mid-point of P_1P_2. Hence prove that, if three points, one on each side of a triangle, are collinear, the three points, one on each side, which are at the same distances as the given points from the mid-points of the sides, are collinear.

CHAPTER VI

CROSS RATIO

1. Cross Ratio of Four Lines. Characteristic of a harmonic set of four lines, L_1, L_2 and L_3, L_4, is that the quotient of the algebraic ratios in which L_3 and L_4 divide L_1, L_2 is -1. We shall now consider this quotient for any four concurrent lines, without restriction as to the value it may assume. The quotient is then known as a *cross ratio*, or anharmonic ratio, of the four lines. We shall denote it by (L_1L_2, L_3L_4).

DEFINITION. *If L_1, L_2, L_3, L_4 are four distinct concurrent lines, of which L_1 and L_2 are finite, the cross ratio (L_1L_2, L_3L_4) is defined as the quotient:*

$$(1) \qquad (L_1L_2, L_3L_4) = \frac{\textit{Ratio in which } L_3 \textit{ divides } L_1, L_2}{\textit{Ratio in which } L_4 \textit{ divides } L_1, L_2}.$$

Let L_1, L_2, L_3, L_4 have respectively the homogeneous coordinates a, b, $a + \lambda_1 b$, $a + \lambda_2 b$. The ratios, μ_1 and μ_2, in which L_3 and L_4 divide L_1, L_2 are, according to Ch. IV, § 3, Th. 1,

$$\mu_1 = -\lambda_1\sqrt{\frac{b_1^2 + b_2^2}{a_1^2 + a_2^2}}, \qquad \mu_2 = -\lambda_2\sqrt{\frac{b_1^2 + b_2^2}{a_1^2 + a_2^2}}.$$

Hence, the cross ratio is

$$(L_1L_2, L_3L_4) = \frac{\mu_1}{\mu_2} = \frac{\lambda_1}{\lambda_2}.$$

THEOREM 1. *If L_1, L_2, L_3, L_4 have respectively the coordinates a, b, $a + \lambda_1 b$, $a + \lambda_2 b$, then*

$$(2) \qquad (L_1L_2, L_3L_4) = \frac{\lambda_1}{\lambda_2}, \qquad \lambda_1\lambda_2(\lambda_1 - \lambda_2) \neq 0.$$

Since the definition prescribes that the four lines be distinct, λ_1 and λ_2 cannot be zero or equal. Hence the cross ratio can never be 0 or 1.

The content of the theorem we take as the definition of the cross ratio in case one of the lines L_1, L_2 is the line at infinity. The following theorems are then universally valid.

THEOREM 2. *The pairs of concurrent lines L_1, L_2 and L_3, L_4 form a harmonic set if and only if*

$$(L_1L_2, L_3L_4) = -1.$$

THEOREM 3. *Interchanging the two pairs of lines does not change the cross ratio:*

$$(L_3L_4,\ L_1L_2) = (L_1L_2,\ L_3L_4).$$

In proving the latter theorem, we take the coordinates of the four lines as in Th. 1. To reverse the rôles of the two pairs, we set

$$a' = a + \lambda_1 b, \qquad b' = a + \lambda_2 b.$$

Then,

$$(\lambda_1 - \lambda_2)\, a = -\lambda_2 a' + \lambda_1 b', \qquad (\lambda_1 - \lambda_2)\, b = a' - b'.$$

Thus, L_3, L_4, L_1, L_2 have respectively the coordinates

$$a', \qquad b', \qquad a' - \frac{\lambda_1}{\lambda_2} b', \qquad a' - b'.$$

Hence, by Th. 1,

$$(L_3L_4,\ L_1L_2) = \frac{-\dfrac{\lambda_1}{\lambda_2}}{-1} = \frac{\lambda_1}{\lambda_2}\,.$$

THEOREM 4. *If three of the four lines and the value of the cross ratio* ($\neq 0, 1$) *are given, the fourth line is uniquely determined.*

For, if L_1, L_2 are given, the third known line determines one of the constants λ_1, λ_2 whose quotient is, by (2), the given cross ratio, so that the other constant can be computed; the fourth line is then uniquely determined. On the other hand, if L_3, L_4 are given, we have but to appeal to Th. 3 and reverse the order of the two pairs of lines.

If the cross ratio defined by (1) is negative, the lines L_3 and L_4 lie in different pairs of regions bounded by L_1 and L_2 (Fig. 1). It is then impossible to rotate a line of one pair *continuously* about C into coincidence with the other line of this pair without passing through one of the lines of the second pair. We express this fact by saying that the two pairs of lines separate one another.

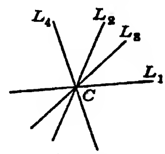


FIG. 1

If the cross ratio is positive, L_3 and L_4 lie in the same pair of regions bounded by L_1 and L_2. In this case, the two pairs of lines do not separate one another.

THEOREM 5. *The pairs of lines L_1, L_2 and L_3, L_4 separate or do not separate one another according as the cross ratio $(L_1L_2,\ L_3L_4)$ is negative or positive.**

* The theorem has been proved only for the case in which the four lines intersect in a finite point. If the lines are parallel, their cross ratio is equal to the cross ratio of the points in which a common perpendicular intersects them. Hence, the theory of separation in this case, and also in the case in which three of the lines are parallel and the fourth is the ideal line, reduces to the corresponding theory for four collinear points. This is discussed in detail in the next paragraph.

EXERCISES

1. Find (L_1L_2, L_3L_4) if L_1, L_2, L_3, L_4 are respectively

(a) $x - y = 0, \quad 2x + y = 0, \quad x + y = 0, \quad 3x - y = 0;$

(b) $2x_1 - x_2 + x_3 = 0, \; 3x_1 + x_2 - 2x_3 = 0, \; 7x_1 - x_2 = 0, \; 5x_1 - x_3 = 0.$

2. If L_1, L_3, L_4 have respectively the equations $2x_1 + x_2 - x_3 = 0$, $x_1 - x_2 + x_3 = 0$, $x_1 = 0$, and $(L_1L_2, L_3L_4) = -2/3$, find the equation of L_2.

2. Cross Ratio of Four Points. The theory in this case is essentially the same as that for lines.

DEFINITION. *If P_1, P_2, P_3, P_4 are four distinct collinear points, of which P_1 and P_2 are finite, the cross ratio (P_1P_2, P_3P_4) is defined as the quotient:*

$$(1) \quad (P_1P_2, P_3P_4) = \frac{\textit{Ratio in which } P_3 \textit{ divides } P_1P_2}{\textit{Ratio in which } P_4 \textit{ divides } P_1P_2} = \frac{\overline{P_3P_1}}{\overline{P_3P_2}} \Big/ \frac{\overline{P_4P_1}}{\overline{P_4P_2}}.$$

From Ch. IV, § 1, Th. 2 we conclude:

THEOREM 1. *If P_1, P_2, P_3, P_4 have the coordinates $a, b, a + \lambda_1 b, a + \lambda_2 b$, then*

$$(2) \qquad (P_1P_2, P_3P_4) = \frac{\lambda_1}{\lambda_2}, \qquad \lambda_1\lambda_2(\lambda_1 - \lambda_2) \neq 0.$$

The content of the theorem we take as the definition of the cross ratio in case one or both of the points P_1, P_2 are ideal.

THEOREM 2. *The pairs of collinear points P_1, P_2 and P_3, P_4 form a harmonic set if and only if $(P_1P_2, P_3P_4) = -1$.*

THEOREM 3. *Interchanging the two pairs of points does not change the value of the cross ratio.*

THEOREM 4. *If three of the four points and the value of the cross ratio ($\neq 0, 1$) are given, the fourth point is uniquely determined.*

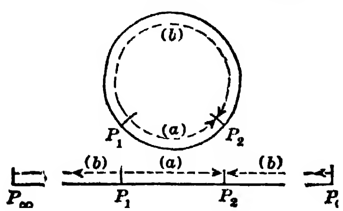


FIG. 2

If P_1 and P_2 are two points on a circle, there are two *continuous* paths along the circle from P_1 to P_2, the paths (a) and (b) in Fig. 2. Similarly, when two points on an extended straight line (a straight line inclusive of its ideal point) are given, there are two *continuous* paths along the line from one point to the other. In the figure, the one path (a) from P_1 to P_2 is along the line-segment P_1P_2, whereas the other path

(b) extends to the left from P_1 through the ideal point P_∞ and thus back to P_2.*

If the cross ratio (P_1P_2, P_3P_4) is negative, one of the points P_3, P_4 lies on the one path, and the other on the second path, from P_1 to P_2, so that it is impossible to proceed continuously from P_1 to P_2 along either path without passing through P_3 or P_4. Similarly, when the rôles of the two pairs of points are reversed. We say, then, that the two pairs separate one another.

If the cross ratio (P_1P_2, P_3P_4) is positive, the points P_3, P_4 both lie on the same path from P_1 to P_2, and the two pairs of points do not separate one another.

THEOREM 5. *The pairs of points P_1, P_2 and P_3, P_4 separate or do not separate one another according as the cross ratio (P_1P_2, P_3P_4) is negative or positive.*†

EXERCISES

1. If P_1, P_2, P_3, P_4 have the coordinates (1, 2), (2, 3), (5, 6), (− 2, − 1), find the cross ratio (P_1P_2, P_3P_4), (a) by finding the ratios in which P_3 and P_4 divide P_1P_2; (b) by finding the ratios in which P_1 and P_2 divide P_3P_4; (c) by expressing the homogeneous coordinates of P_3 and P_4 as linear combinations of those of P_1 and P_2.

2. If P_1, P_2, P_4 have the coordinates (1, 1, 1), (1, − 1, 1), (1, 0, 1) and $(P_1P_2, P_3P_4) = 2$, find the coordinates of P_3.

3. Show that, if P_4 is the point at infinity on the line of the finite distinct points P_1, P_2, P_3, the cross ratio (P_1P_2, P_3P_4) is equal to the ratio in which P_3 divides the line-segment P_1P_2.

3. Cross Ratio a Projective Property. The crux of the proof that cross ratio, even though originally defined in terms of distances, is preserved by all projections consists in establishing the following theorem.

THEOREM 1. *If P_1P_2, P_3P_4 are four distinct collinear points lying respectively on the four distinct concurrent lines L_1, L_2, L_3, L_4, then*

$$(L_1L_2, L_3L_4) = (P_1P_2, P_3P_4).$$

* The question of the continuity of path (b) is discussed in detail in Ch. IX, § 1.

† If P_1, P_2, P_3, P_4 are points at infinity, the cross ratio (P_1P_2, P_3P_4) is equal to the cross ratio (L_1L_2, L_3L_4) of the lines L_1, L_2, L_3, L_4 joining P_1, P_2, P_3, P_4 to a finite point, as will be proved in the next paragraph. Consequently, if we agree that the pairs of points P_1, P_2 and P_3, P_4 separate or do not separate one another according as the pairs of lines L_1, L_2 and L_3, L_4 separate or do not separate one another, Theorem 5 is valid without exception.

The theorem may be established by the method of Ch. IV, § 5. By means of it we can prove, after the manner of the corresponding proof in the case of harmonic division, that cross ratio is a projective property. We leave the details to the reader.

The dual of four distinct collinear points is four distinct concurrent lines. We now agree, further, that the dual of the fact that the four points, in a given order, have a certain cross ratio, is that the four lines, in the corresponding order, have the same cross ratio, and vice versa.

4. Formulas for Cross Ratio. The following table gives the values of the cross ratio (E_1E_2, E_3E_4) for various forms of the coordinates of four distinct elements, E_1, E_2, E_3, E_4, of a range of points or a pencil of lines.

	E_1	E_2	E_3	E_4	(E_1E_2, E_3E_4)
(1)	a	b	$a + \lambda_1 b$	$a + \lambda_2 b$	$\dfrac{\lambda_1}{\lambda_2}$
(2)	a	b	$k_1a + l_1b$	$k_2a + l_2b$	$\dfrac{l_1k_2}{k_1l_2}$
(3)	$a + \lambda_1 b$	$a + \lambda_2 b$	$a + \lambda_3 b$	$a + \lambda_4 b$	$\dfrac{(\lambda_3 - \lambda_1)(\lambda_4 - \lambda_2)}{(\lambda_3 - \lambda_2)(\lambda_4 - \lambda_1)}$
(4)	$k_1a + l_1b$	$k_2a + l_2b$	$k_3a + l_3b$	$k_4a + l_4b$	$\dfrac{\lvert k_3\,l_1\rvert \cdot \lvert k_4\,l_2\rvert}{\lvert k_3\,l_2\rvert \cdot \lvert k_4\,l_1\rvert}$.

In case (4), $|k_3\,l_1|$, for example, is the determinant $k_3l_1 - k_1l_3$.

Case (1) has already been treated. The value of the cross ratio in (2) is found by applying (1). In the same way, (4) follows from (3).

To establish (3), rewrite the coordinates of the elements in the form (2). Set

$$a + \lambda_1 b = a', \qquad a + \lambda_2 b = b',$$

solve these symbolic equations for a and b in terms of a' and b', and thus express the coordinates of E_3 and E_4 as linear combinations of a' and b'. Coordinates of the four elements, thus obtained, are

$$a', \quad b', \quad (\lambda_3 - \lambda_2)\,a' - (\lambda_3 - \lambda_1)\,b', \quad (\lambda_4 - \lambda_2)\,a' - (\lambda_4 - \lambda_1)\,b'.$$

Application of (2) yields the result listed in (3).

EXERCISES

1. Deduce the value of the cross ratio in (4).

2. Show that if x_1, x_2, x_3, x_4 are the abscissas of four distinct points on the axis of x,

$$(P_1P_2, P_3P_4) = \frac{(x_3 - x_1)(x_4 - x_2)}{(x_3 - x_2)(x_4 - x_1)}.$$

3. The slopes of four distinct lines passing through a finite point are λ_1, λ_2, λ_3, λ_4. Prove that

$$(L_1L_2, L_3L_4) = \frac{(\lambda_3 - \lambda_1)(\lambda_4 - \lambda_2)}{(\lambda_3 - \lambda_2)(\lambda_4 - \lambda_1)}.$$

4. If (x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4) are four distinct collinear points, show that

$$(P_1P_2, P_3P_4) = \frac{(x_3 - x_1)(x_4 - x_2)}{(x_3 - x_2)(x_4 - x_1)},$$

provided the line of the four points is not parallel to the y-axis. What does the result become if homogeneous coordinates, $a : (a_1, a_2, a_3)$, b, c, d are introduced for the four points?

5. Prove that, if a, b, c, d are four distinct lines through a point which does not lie on the axis of x, their cross ratio, in the order given, is

$$\frac{|c_1 a_3| \cdot |d_1 b_3|}{|c_1 b_3| \cdot |d_1 a_3|},$$

where $|c_1 a_3|$, for example, is $c_1a_3 - c_3a_1$.

6. Four points a, b, c, d, not necessarily collinear, determine with a fifth point r four distinct lines L_1, L_2, L_3, L_4. Show that

$$(L_1L_2, L_3L_4) = \frac{|c\,a\,r| \cdot |d\,b\,r|}{|c\,b\,r| \cdot |d\,a\,r|}.$$

Suggestion. Make use of Ch. V, § 4, Ex. 6.

7. Show that, if we give to each of four concurrent lines L_1, L_2, L_3, L_4 a sense and understand by (L_3L_1), for example, the directed angle through which L_3 must be rotated in order that it coincide in position and sense with L_1, then

$$(L_1L_2, L_3L_4) = \frac{\sin(L_3L_1)\sin(L_4L_2)}{\sin(L_3L_2)\sin(L_4L_1)}.$$

It is assumed, of course, that the lines concur in a finite point.

8. Prove that the cross ratio of the four lines joining four given points on a circle to any fifth point on the circle is constant.

5. The Twenty-Four Cross Ratios of Four Elements. Four distinct elements, four collinear points or four concurrent lines, can be arranged in twenty-four orders. Consequently, they have twenty-four cross ratios. We proceed to discuss the relationships between these cross ratios.

It is natural to pick out one cross ratio and determine the relations of the others to it. Arrange the four elements in an arbitrary but specific order and call them, in this order, E_1, E_2, E_3, E_4. If coordinates for them are taken in the forms

$$a + \lambda_1 b, \qquad a + \lambda_2 b, \qquad a + \lambda_3 b, \qquad a + \lambda_4 b,$$

the cross ratio $(E_1E_2,\ E_3E_4)$, which we shall now write more simply as $(1\ 2,\ 3\ 4)$, has the value

$$(1)\qquad (1\ 2,\ 3\ 4) = \frac{(\lambda_3 - \lambda_1)(\lambda_4 - \lambda_2)}{(\lambda_3 - \lambda_2)(\lambda_4 - \lambda_1)}\,.$$

We have already proved

THEOREM 1. *Interchanging the two pairs of elements does not change the cross ratio:*

$$(3\ 4,\ 1\ 2) = (1\ 2,\ 3\ 4).$$

Our experience with harmonic sets suggests

THEOREM 2. *The reversal of the order in both pairs does not change the cross ratio:*

$$(2\ 1,\ 4\ 3) = (1\ 2,\ 3\ 4).$$

The theorem is true, for if in (1) we interchange 1 and 2, and also 3 and 4, the result is

$$(2\ 1,\ 4\ 3) = \frac{(\lambda_4 - \lambda_2)(\lambda_3 - \lambda_1)}{(\lambda_4 - \lambda_1)(\lambda_3 - \lambda_2)} = (1\ 2,\ 3\ 4).$$

It now follows that the twenty-four cross ratios fall into six sets of four each, so that the four cross ratios in a set are all equal. For example,

$$(2)\qquad (1\ 2,\ 3\ 4) = (3\ 4,\ 1\ 2) = (2\ 1,\ 4\ 3) = (4\ 3,\ 2\ 1);$$

$$(3)\qquad (2\ 1,\ 3\ 4) = (3\ 4,\ 2\ 1) = (1\ 2,\ 4\ 3) = (4\ 3,\ 1\ 2).$$

THEOREM 3. *The reversal of the order in one pair changes the cross ratio to its reciprocal:*

$$(2\ 1,\ 3\ 4) = \frac{1}{(1\ 2,\ 3\ 4)}\,.$$

The proof of the theorem is similar to that of Theorem 2.

Thus far, we have kept the pairs intact. Let us now split them and consider, for example, $(1\ 3,\ 2\ 4)$ and $(1\ 4,\ 2\ 3)$. It is readily found that, if we denote the value of $(1\ 2,\ 3\ 4)$ by α:

$$(1\ 2,\ 3\ 4) = \alpha,$$

then

$$(4) \qquad (1\,3,\,2\,4) = 1 - \alpha, \qquad (1\,4,\,2\,3) = \frac{\alpha - 1}{\alpha}.$$

These results, in conjunction with Theorem 3, enable us to write down the value of a representative cross ratio of each of the six sets:

$$(5) \qquad \begin{array}{lll} (1\,2,\,3\,4) = \alpha, & (1\,3,\,2\,4) = 1 - \alpha, & (1\,4,\,2\,3) = \dfrac{\alpha - 1}{\alpha}, \\ (2\,1,\,3\,4) = \dfrac{1}{\alpha}, & (3\,1,\,2\,4) = \dfrac{1}{1 - \alpha}, & (4\,1,\,2\,3) = \dfrac{\alpha}{\alpha - 1}. \end{array}$$

For example, the four cross ratios in (2) all have the value α and the four in (3) the value $1/\alpha$.

Suppose that the four elements form a harmonic set when E_1 is paired with E_2 and E_3 with E_4: $(E_1E_2,\, E_3E_4) = -1$. Then the six cross ratios in (5) are found to be equal in pairs, to -1, 2, and 1/2, respectively. Conversely, if α has one of these three values, two of the cross ratios in (5) are -1 and the four elements, properly paired, form a harmonic set.

Theorem 4. *A necessary and sufficient condition that four elements can be so paired that they form a harmonic set is that a random cross ratio have one of the values,* -1, 2, *or* 1/2.

It is only when the four elements, properly paired, form a harmonic set that the six cross ratios in (5) fail to be distinct. For, it is readily shown that the only *real* solutions, other than $\alpha = 0$ or 1, of the equations obtained by equating the value, α, of the first cross ratio to that of each of the other five in turn, are $\alpha = -1$, 2, and 1/2. It is unnecessary to equate the values of each two cross ratios, inasmuch as (1 2, 3 4) was an arbitrary cross ratio of the original twenty-four.

We summarize our results in the following theorem.

Theorem 5. *The twenty-four cross ratios fall into six sets of four each. The four cross ratios of a set have the same value. The six values are all distinct unless the four elements, properly paired, form a harmonic set.*

EXERCISES

1. Establish the results expressed in equations (4).

2. Show that if E_1, E_2, E_3, E_4, E_5 are distinct concurrent lines, or distinct collinear points,

$$(1\,2,\,3\,4)(1\,2,\,4\,5)(1\,2,\,5\,3) = 1.$$

6. Applications of Cross Ratio. The Theorems of Menelaus and Ceva. Let P_1, P_2, P_3 be the vertices of a triangle, and let Q_1, Q_2, Q_3 be three collinear points, distinct from the vertices and lying one on each side, Q_1 on P_2P_3, Q_2 on P_3P_1, and Q_3 on P_1P_2. Mark three other points, Q'_1, Q'_2, Q'_3, one on each side of the triangle and distinct from the vertices, and form the cross ratios:

$$(1)\quad (P_2P_3,\, Q'_1Q_1) = k_1, \qquad (P_3P_1,\, Q'_2Q_2) = k_2, \qquad (P_1P_2,\, Q'_3Q_3) = k_3.$$

THEOREM 1. *A necessary and sufficient condition that the points Q'_1, Q'_2, Q'_3 be collinear is that the product of the three cross ratios be equal to unity: $k_1k_2k_3 = 1$.*

Let P_1, P_2, P_3 have the coordinates a, b, c. By Ch. III, § 7, Ex. 4, the coordinates of Q_1, Q_2, Q_3 can be written in the forms

$$Bb - Cc, \qquad Cc - Aa, \qquad Aa - Bb,$$

where, since Q_1, Q_2, Q_3 are distinct from the vertices, no one of the constants A, B, C is zero. Coordinates for Q'_1, Q'_2, Q'_3 are then found, by taking account of (1), to be

$$Bb - k_1Cc, \qquad Cc - k_2Aa, \qquad Aa - k_3Bb.$$

These points are collinear if and only if three constants l, m, n, not all zero, exist so that

$$l(Bb - k_1Cc) + m(Cc - k_2Aa) + n(Aa - k_3Bb) = 0,$$

or

$$(n - k_2m)Aa + (l - k_3n)Bb + (m - k_1l)Cc = 0.$$

This symbolic equation, since a, b, c are not collinear and $ABC \neq 0$, is equivalent to the three equations

$$\begin{aligned} -k_2m + n &= 0, \\ l \qquad - k_3n &= 0, \\ -k_1l + m \qquad &= 0. \end{aligned}$$

A condition necessary and sufficient that these equations have a solution for l, m, n, not 0, 0, 0, is that the determinant of their coefficients vanish. But this condition reduces immediately to $k_1k_2k_3 = 1$.

THEOREM 2. *The lines, $P_1Q'_1$, $P_2Q'_2$, $P_3Q'_3$, joining the points Q'_1, Q'_2, Q'_3 to the opposite vertices of the triangle are concurrent if and only if the product of the three cross ratios is -1: $k_1k_2k_3 = -1$.*

The proof is left to the reader.

If Q_1, Q_2, Q_3 are the points at infinity on the sides of the triangle, k_1, k_2, k_3 are respectively the algebraic ratios in which Q_1', Q_2', Q_3' divide the sides P_2P_3, P_3P_1, P_1P_2; see § 2, Ex. 3. Theorems 1 and 2 then became the classical theorems of Menelaus and Ceva.

THEOREM OF MENELAUS. *Three points, one on each side of a triangle $P_1P_2P_3$, are collinear if and only if the product of the algebraic ratios in which they divide the sides (P_2P_3, P_3P_1, P_1P_2) is unity.*

THEOREM OF CEVA. *A necessary and sufficient condition that the lines which join three points, one on each side of a triangle, to the opposite vertices be concurrent is that the product of the algebraic ratios in which the three points divide the sides be -1.*

As an application of the Theorem of Menelaus we establish an interesting property of the complete quadrilateral.

THEOREM 3. *The mid-points of the diagonals of a complete quadrilateral (with finite vertices) lie on a line.*

Let P, Q, R be the mid-points of the diagonals and P', Q', R', the mid-points of the sides of the triangle $P_1Q_1R_1$, as shown in the figure. Since R', Q', and P are the mid-points of the lines which join P_1 with Q_1, R_1, and P_2, the point P lies on the line $R'Q'$. Similarly, R lies on $Q'P'$, and Q on $P'R'$. Thus, P, R, Q lie respectively on the sides of the triangle $P'Q'R'$. Therefore, if we can show that

$$\frac{\overline{PR'}\;\overline{RQ'}\;\overline{QP'}}{\overline{PQ'}\;\overline{RP'}\;\overline{QR'}} = 1, \tag{2}$$

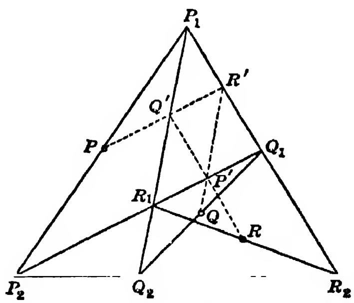


FIG. 3

it will follow that P, Q, R are collinear.

We note that P_2, Q_2, R_2 are collinear points, one on each side of the triangle $P_1Q_1R_1$. Hence

$$\frac{\overline{P_2Q_1}\;\overline{Q_2R_1}\;\overline{R_2P_1}}{\overline{P_2R_1}\;\overline{Q_2P_1}\;\overline{R_2Q_1}} = 1. \tag{3}$$

Since the line $PQ'R'$ is parallel to the line $P_2R_1Q_1$, it follows that

$$\frac{\overline{P_2Q_1}}{\overline{P_2R_1}} = \frac{\overline{PR'}}{\overline{PQ'}}.$$

In the same way,

$$\frac{\overline{Q_2R_1}}{\overline{Q_2P_1}} = \frac{\overline{QP'}}{\overline{QR'}}, \qquad \frac{\overline{R_2P_1}}{\overline{R_2Q_1}} = \frac{\overline{RQ'}}{\overline{RP'}}.$$

Consequently, (3) reduces to (2) and the theorem is proved.

EXERCISES

1. Prove Theorem 2.

2. State the theorem which is the dual of Theorem 1 and show that the proof of Theorem 1 can be reinterpreted as a proof of the dual theorem.

7. Cross Ratios, When the Four Elements are Not Distinct. By means of the formula,

$$(1\,2,\,3\,4) = \frac{(\lambda_3 - \lambda_1)(\lambda_4 - \lambda_2)}{(\lambda_3 - \lambda_2)(\lambda_4 - \lambda_1)},$$

for the cross ratio $(E_1E_2,\ E_3E_4)$ of the four distinct elements $a+\lambda_1 b$, $a + \lambda_2 b$, $a + \lambda_3 b$, $a + \lambda_4 b$, the following limits are readily established:

$$(1) \quad \lim_{E_4 \to E_3} (1\,2,\,3\,4) = 1, \quad \lim_{E_4 \to E_2} (1\,2,\,3\,4) = 0, \quad \lim_{E_4 \to E_1} (1\,2,\,3\,4) = \infty.$$

On the basis of these results we adopt the *definitions:**

$$(2) \quad (E_1E_2,\ E_3E_3) = 1, \qquad (E_1E_2,\ E_3E_2) = 0, \qquad (E_1E_2,\ E_3E_1) = \infty.$$

Definitions covering the remaining cases in which two of the elements are the same may be arrived at by noting that, since Theorems 1, 2 of § 5 are always true for four distinct elements, they remain valid when two of the elements coincide. For example, since $(E_3E_4,\ E_1E_2)$ is equal to $(E_1E_2,\ E_3E_4)$, we define: $(E_3E_3,\ E_1E_2) = 1$.

The recognition of 0 and 1 as values of a cross ratio removes the restrictions from Theorems 4 of §§ 1, 2. We can now say, for example:

THEOREM 1. *If P_1, P_2, P_3 are distinct collinear points, there exists a unique point P on their line for which $(P_1P_2,\ P_3P)$ has a prescribed value, defined or infinite. In particular, according as ∞,† 0, or 1 is prescribed, P is P_1, P_2, or P_3.*

Three of the four elements coincide only when the elements of one pair coincide with one of the elements of the other pair. Let us allow

* Strictly speaking, only the first two of equations (2) represent definitions. We mean by the third: $(E_1E_2,\ E_3E_1) = \infty$, merely the last equation of (1), that is, that $(E_1E_2,\ E_3E_4)$ becomes infinite when E_4 approaches E_1 as a limit.

† This means that when the prescribed value, thought of as variable, becomes infinite, P approaches P_1 as a limit.

E_3 and E_4 to approach E_2 in that we set

$$\lambda_3 = \lambda_2 + \epsilon, \qquad \lambda_4 = \lambda_2 + \eta,$$

and eventually allow both ϵ and η to approach zero. The cross ratio (1 2, 3 4) becomes

$$(1\,2,\,3\,4) = \frac{\lambda_2 - \lambda_1 + \epsilon}{\lambda_2 - \lambda_1 + \eta}\,\frac{\eta}{\epsilon}.$$

The limit of the first quotient, as ϵ and η approach zero, is unity; η/ϵ is, however, the ratio of two independent infinitesimals and can be made to approach any preassigned limit or to become infinite by properly controlling these infinitesimals.

The situation is similar, if we let E_3 and E_4 approach E_1, or if we let three of the elements approach the fourth. Accordingly, we refrain from defining the cross ratio in all these cases.

EXERCISES

1. Establish the limits in (1).
2. By means of limiting processes, justify the definitions:

$$(E_1E_1,\, E_3E_3) = 1, \qquad (E_1E_2,\, E_1E_2) = 0, \qquad (E_1E_2,\, E_2E_1) = \infty\,.$$

CHAPTER VII

TRANSFORMATIONS

Rigid motions and projections are examples of operations which are technically known as transformations. We have recognized their importance as a basis for classifying geometric properties. We proceed now to study them intensively.

A. Rigid Motions *

1. Simple Rigid Motions of the Plane. *Translations.* A translation of a plane consists in moving each point in the plane a given distance in a given direction. It is completely determined by a directed line-segment, $\overline{AA'}$, in the plane, the direction of $\overline{AA'}$ giving the direction of the translation and the length of $\overline{AA'}$ the amount. If the projections of $\overline{AA'}$ on the axes are a and b, and the translation carries the arbitrary point $P : (x, y)$ into the point $P' : (x', y')$, then

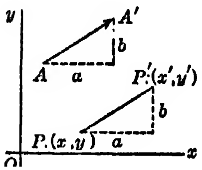


Fig. 1

(1) $$x' = x + a, \qquad y' = y + b.$$

These equations are called *the equations of the translation.*

Rotations. We consider first the rotation of the plane about the origin through the algebraic angle θ. Let the rotation carry the point $P : (x, y)$ into the point $P' : (x', y')$ and the projection, M, of P on the axis of x into the point M'. Then $\overline{OM} = \overline{OM'} = x$ and $\overline{MP} = \overline{M'P'} = y$. Evidently,

$$\text{Proj } OP' = \text{Proj } \overline{OM'} + \text{Proj } M'P'.$$

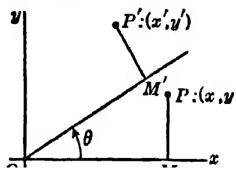


Fig. 2

Taking the projections, first on the axis of x and then on the axis of y, we obtain as the equations of the rotation

(2) $$x' = x \cos\theta - y \sin\theta, \qquad y' = x \sin\theta + y \cos\theta.$$

In a similar manner the equations of the rotation about the point

* In this part of the chapter we shall restrict ourselves to the *finite* plane.

(x_0, y_0) through the angle θ are found to be

$$(3) \qquad \begin{aligned} x' - x_0 &= (x - x_0)\cos\theta - (y - y_0)\sin\theta, \\ y' - y_0 &= (x - x_0)\sin\theta + (y - y_0)\cos\theta. \end{aligned}$$

Example. Let it be required to find the equation of the parabola C', with vertex at the origin, latus rectum of length 2, and axis inclined at an angle of 30° to the axis of x, as shown in Fig. 3.

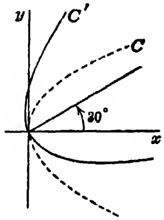


FIG. 3

The parabola C' can be obtained by rotating the congruent parabola C of the figure about the origin through 30°. The equation of C is

$$y^2 = 2x.$$

The rotation is

$$x' = \tfrac{1}{2}\sqrt{3}\,x - \tfrac{1}{2}\,y, \qquad y' = \tfrac{1}{2}\,x + \tfrac{1}{2}\sqrt{3}\,y.$$

Solving these equations for x, y in terms of x', y':

$$x = \tfrac{1}{2}\sqrt{3}\,x' + \tfrac{1}{2}\,y', \qquad y = -\tfrac{1}{2}\,x' + \tfrac{1}{2}\sqrt{3}\,y',$$

and substituting in the equation of C, we obtain the equation of C':

$$(-\tfrac{1}{2}\,x' + \tfrac{1}{2}\sqrt{3}\,y')^2 - 2(\tfrac{1}{2}\sqrt{3}\,x' + \tfrac{1}{2}\,y') = 0,$$

or

$$x'^2 - 2\sqrt{3}\,x'y' + 3\,y'^2 - 4\sqrt{3}\,x' - 4\,y' = 0.$$

EXERCISES

1. Show that there is a unique translation which carries a given point into a prescribed point. Find the equations of the translation which carries the point (2, 3) into the point (0, − 1). Apply this translation to the curve

$$y^2 - x - 8y + 18 = 0.$$

2. Show that there is a unique rotation about a point P_0 which carries a given point A into a prescribed point A', provided $P_0A = P_0A'$. Find the equations of the rotation about the origin which carries the point (3, 1) into the point (− 1, 3).

3. Rotate the hyperbola $x^2 - y^2 = a^2$ through 45° about the origin.

4. Find the equation of the ellipse, center at the origin, with semi-axes 3 and 2, and with the line $x - 2y = 0$ as transverse axis.

5. Prove both geometrically and analytically that, if an arbitrary line is rotated about a point through the angle θ, the angle from this line to the new line is equal to θ.

2. Product of Two Transformations. Inverse of a Transformation.

Example. Find the equation of the ellipse C' whose center is at the point (4, 6), whose semi-axes are of lengths 3 and 2, and whose transverse axis has the slope unity.

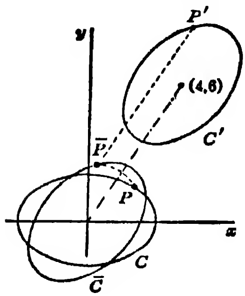


FIG. 4

Congruent to the ellipse C' is the ellipse

$$C: \qquad 4x^2 + 9y^2 = 36,$$

with center at the origin and axis of x as transverse axis. To obtain the equation of C', we shall rotate C about O through 45° into the ellipse $\bar{C}$ and then translate $\bar{C}$ into C', as indicated in Fig. 4.

The equations of the rotation carrying $P : (x, y)$ into $\bar{P} : (\bar{x}, \bar{y})$ are

$$T_1: \qquad \bar{x} = \tfrac{1}{2}\sqrt{2}\,(x - y), \qquad \bar{y} = \tfrac{1}{2}\sqrt{2}\,(x + y).$$

Those of the translation carrying $\bar{P} : (\bar{x}, \bar{y})$ into $P' : (x', y')$ are

$$T_2: \qquad x' = \bar{x} + 4, \qquad y' = \bar{y} + 6.$$

By the methods of § 1 we readily show that the rotation T_1 carries the ellipse C into

$$\bar{C}: \qquad 4(\bar{x} + \bar{y})^2 + 9(\bar{x} - \bar{y})^2 = 72,$$

and that the translation T_2 carries $\bar{C}$ into

$$C': \qquad 4(x' + y' - 10)^2 + 9(x' - y' + 2)^2 = 72.$$

Expanding and collecting terms, we obtain as the required equation

$$13\,x'^2 - 10\,x'y' + 13\,y'^2 - 44\,x' - 116\,y' + 364 = 0.$$

Alternative Procedure. If we substitute for $\bar{x}$, $\bar{y}$ in T_2 their values from T_1, we obtain a transformation T,

$$T: \qquad x' = \tfrac{1}{2}\sqrt{2}\,(x - y) + 4, \qquad y' = \tfrac{1}{2}\sqrt{2}\,(x + y) + 6,$$

which carries $P : (x, y)$ directly into $P' : (x', y')$ and hence should carry C directly into C'. As a matter of fact, if we solve the equations of T for x, y:

$$x = \tfrac{1}{2}\sqrt{2}\,(x' + y' - 10), \qquad y = -\tfrac{1}{2}\sqrt{2}\,(x' - y' + 2),$$

and substitute in the equation of C, the result is the equation of C'.

Product of Two Transformations. The two procedures may be represented by the following diagrams:

$$P \xrightarrow{T_1} \bar{P} \xrightarrow{T_2} P'; \qquad P \xrightarrow{\qquad T \qquad} P'.$$

Evidently, T is the transformation which results from carrying out successively the transformations T_1 and T_2. Technically, it is known as *the product of these transformations.* We express its relationship to them by writing, symbolically,

$$T = T_1T_2.$$

The product of two numbers is commutative, that is, independent of the order of the factors. Is this also true of transformations? Will the transformation T_1T_2, the result of following T_1 by T_2, be the same as the transformation T_2T_1, the result of following T_2 by T_1?

Suppose that, in our example, we first translate C into $\bar{C}$ and then rotate $\bar{C}$ into C'', using the same translation and rotation as before. Will C'' be the same as C'? Even a rough figure will give the answer: No!

Analytically, we have as the translation T_2, carrying (x, y) into $(\bar{x}, \bar{y})$:

$$\bar{x} = x + 4, \qquad \bar{y} = y + 6,$$

and as the rotation T_1, carrying $(\bar{x}, \bar{y})$ into (x'', y''):

$$x'' = \tfrac{1}{2}\sqrt{2}\,(\bar{x} - \bar{y}), \qquad y'' = \tfrac{1}{2}\sqrt{2}\,(\bar{x} + \bar{y}).$$

Eliminating the intermediate variables $\bar{x}$, $\bar{y}$, we obtain as the product T_2T_1:

$$x'' = \tfrac{1}{2}\sqrt{2}\,(x - y - 2), \qquad y'' = \tfrac{1}{2}\sqrt{2}\,(x + y + 10).$$

Evidently, this product T_2T_1 is different from the previous product T_1T_2.

The result here is typical. *The product of two transformations is, in general, not commutative.*

Inverse of a Transformation. The inverse of a transformation T is the transformation which undoes the work of T, in that it carries all points back into their original positions. Thus, the inverse of the translation defined by a directed line-segment is the translation defined by the same line-segment oppositely directed.

If one transformation is the inverse of a second, the second transformation is the inverse of the first. The two transformations are reciprocally related. This relationship is analogous to that of a number, not zero, and its reciprocal. Accordingly, it is natural to denote the inverse of a transformation T by the symbol T^{-1}. To express the fact that the inverse of T^{-1} is T, we have $(T^{-1})^{-1} = T$, an equation which is actually in agreement with the laws of algebra.

The product of two numbers which are reciprocals of one another is unity. Similarly, since each of two transformations which are inverses of one another undoes the work of the other, their product, in either order, is *the identical transformation* I,

$$x' = x, \qquad y' = y,$$

which leaves each point unmoved. Symbolically,

$$TT^{-1} = I \qquad \text{or} \qquad T^{-1}T = I.$$

Here again, if we think of I as symbolic for unity, the laws of algebra are preserved.

If a transformation T carries the arbitrary point $P : (x, y)$ into the point $P' : (x', y')$, the inverse of T carries $P' : (x', y')$ back into $P : (x, y)$. Hence, the equations of the inverse of T can be found, in general, by solving the equations of T for x, y in terms of x', y'. For example, if T is the rotation

$$T: \qquad x' = x \cos \theta - y \sin \theta, \qquad y' = x \sin \theta + y \cos \theta,$$

the inverse T^{-1}, found in this way, is

$$T^{-1}: \qquad x = x' \cos \theta + y' \sin \theta, \qquad y = -x' \sin \theta + y' \cos \theta.$$

EXERCISES

1. If T_1 and T_2 are $x' = x - 2$, $y' = y + 1$ and $x' = -y$, $y' = x$, find the products T_1T_2 and T_2T_1.

2. Verify, analytically, that $TT^{-1} = I$ in the case of the rotation about O through the angle θ.

3. Find the equation of the hyperbola which has its center at $(1, -2)$, major and minor axes of lengths 4 and 6, and transverse axis of slope $-3/4$.

4. *Associative Law for the Product of Transformations.* The product of three transformations T_1, T_2, T_3 in the order given can be found in two ways, which are symbolized by $(T_1T_2)T_3$ and $T_1(T_2T_3)$. In the first case, we form T_1T_2 and then take the product of T_1T_2 and T_3; in the second, we form T_2T_3 and then take the product of T_1 and T_2T_3. The two transformations which result are always the same:

$$(T_1T_2)T_3 = T_1(T_2T_3),$$

and hence can be denoted simply by $T_1T_2T_3$.

Verify the law when T_1 and T_2 are the transformations of Ex. 1 and T_3 is the translation: $x' = x + 3$, $y' = y - 2$.

5. Show that the product of two translations is always commutative.

6. Show that the product of two rotations about the origin is always commutative.

7. Is the product of a translation and a rotation ever commutative?

3. The General Rigid Motion of the Plane. We inquire whether, in the translations and rotations, we have all the rigid motions of the plane.

We consider, first, the number of pairs of corresponding points necessary to determine completely a rigid motion. One pair, A, A', is not enough; for, after the plane has been moved so that A coincides with A', it can still be rotated at will about A'.

Two pairs, A, A' and B, B', chosen so that the corresponding distances AB and $A'B'$ are equal, are enough; for, once the plane has been moved rigidly so that A and B coincide respectively with A' and B', no further motion is possible.*

THEOREM 1. *There exists a unique rigid motion which carries two given points A, B into two specified points A', B', provided that the distance $A'B'$ is equal to the distance AB.*

Let us now consider an arbitrary rigid motion S, and think of it as determined by two pairs of corresponding points, A, A' and B, B', where $AB = A'B'$.

We can show that S is, for example, a rotation, if we can exhibit a rotation R which carries A into A' and B into B'; for, S and R would both carry A into A' and B into B', and hence, by Th. 1, would be identical.

If there exists a rotation carrying A into A' and B into B', its center must be in the point C (Fig. 5) in which the perpendicular bisectors,

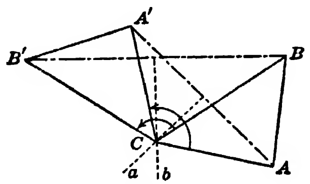


FIG. 5

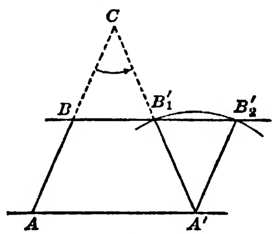


FIG. 6

a and b, of AA' and BB' intersect. The existence of the rotation is, then, assured if the angles ACA' and BCB' are equal, both in mag-

* In this connection it is well to emphasize the fact that a transformation takes into account only the initial and final positions of the points in the plane. The paths from the initial to the final positions are immaterial. In the case of the rigid motion which carries A into A' and B into B', there are infinitely many ways in which the plane can be moved from the initial to the final position. The rigid motion, however, is unique, since the final position of the plane is uniquely determined.

nitude and sense. That this is true follows immediately from the fact that the triangles ACB and $A'CB'$ have their corresponding sides equal. Hence S is a rotation.

We assumed in this discussion that a and b, and hence AA' and BB', were not parallel or identical. Suppose that AA' and BB' are parallel. Mark A, A', and B, as shown in Fig. 6. Since $A'B'=AB$, B' can have either of the two positions, B_1' and B_2'.* If B' is in the position B_1', S is the rotation about C through the angle indicated. If B' is in the position B_2', S is the translation represented by the directed line-segment $\overline{AA'}$.

The case in which the lines AA' and BB' are identical lends itself to similar treatment. The details are left to the reader.

We have now considered all possibilities, and have found that S is always either a rotation or a translation.

THEOREM 2. *A rigid motion is either a translation or a rotation.*

All rigid motions are, therefore, given by equations (1) and (3) of § 1. To obtain a single pair of equations representing them all, we proceed in a slightly different fashion.

Consider once more the arbitrary rigid motion S. If AB and $A'B'$ are parallel and have the same sense, S is evidently a translation.

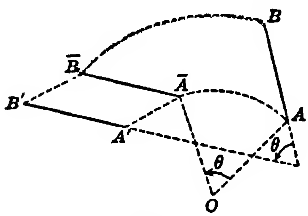


FIG. 7

Suppose that AB and $A'B'$ are not parallel with the same sense. Let θ be the angle from the line AB, directed from A to B, to the line $A'B'$, directed from A' to B', and let the rotation R about O through the angle θ carry A and B into $\bar{A}$ and $\bar{B}$. Either $\bar{A}$ and $\bar{B}$ coincide respectively with A' and B', or $\bar{A}\bar{B}$ and $A'B'$ are parallel and have the same sense; see § 1, Ex. 5. In the first case, S is the rotation R; in the second, S is the rotation R followed by the translation which carries $\bar{A}$, $\bar{B}$ into A', B'.

THEOREM 3. *A rigid motion is a rotation about the origin followed by a translation, or either alone.*

The product of the general rotation about O and the general translation:

$$\bar{x} = x\cos\theta - y\sin\theta, \qquad x' = \bar{x} + a,$$
$$\bar{y} = x\sin\theta + y\cos\theta, \qquad y' = \bar{y} + b,$$

* Unless AB is perpendicular to AA'. What is S in this case?

has the equations

$$(1) \qquad \begin{aligned} x' &= x \cos \theta - y \sin \theta + a, \\ y' &= x \sin \theta + y \cos \theta + b. \end{aligned}$$

These equations, moreover, include the rotations about O, when $a = b = 0$, and the translations, when $\theta = 0$. Hence

THEOREM 4. *All the rigid motions of the plane are represented by equations* (1).

EXERCISES

1. Find the equations of the rigid motion which carries the points (4, 2) and (3, 5) respectively into the points (2, 4) and ($-$ 1, 5). Use the method of Fig. 5.

2. Show that the rigid motion $x' = -y + 3$, $y' = x + 1$ is a rotation about the point (1, 2).

3. Express the rigid motion of Ex. 2 as the product of a rotation about the origin and a translation.

4. Groups of Transformations. The word "group" in the theory of transformations has a highly specialized meaning. We define it as follows.

DEFINITION. *A set of transformations, finite or infinite in number, is called a group, if*

(*a*) *the product of every two transformations of the set (including, in particular, the product of every transformation with itself) belongs to the set; and*

(*b*) *the inverse of every transformation of the set belongs to the set.*

The identity and the rotation about the origin through 180°,

$$(1) \qquad x' = x, \quad y' = y \qquad \text{and} \qquad x' = -x, \quad y' = -y,$$

form a group. For, each is its own inverse, the product of each with itself is the identity, and the product of the two is the rotation.

Similarly, the four transformations

$$(2) \qquad \begin{matrix} x' = x, & x' = -y, & x' = -x, & x' = y, \\ y' = y, & y' = x, & y' = -y, & y' = -x, \end{matrix}$$

namely, the identity and the three rotations about the origin through 90°, 180°, and 270°, respectively, form a group.

Groups (1) and (2) both contain the identical transformation. So, also, does every group. For, if T is a transformation of a given group, its inverse T^{-1} belongs to the group, and hence the product TT^{-1} belongs to the group; but this product is the identical transformation.

THEOREM 1. *Every group of transformations contains the identity.*

The group (1) is a *subgroup* of the group (2) in that its transformations are included among those of (2). In general, if among the transformations of a given group there exists a subset of transformations which itself forms a group, this group is called a subgroup of the given group.

The Group of Rigid Motions and Its Subgroups. Consider the set of all rigid motions of the plane,

$$(3) \quad x' = x \cos\theta - y \sin\theta + a, \qquad y' = x \sin\theta + y \cos\theta + b,$$

where a, b, and θ are arbitrary constants. The product of any two of them, since it is necessarily a rigid motion, belongs to the set. For the same reason, the inverse of every one of them is found in the set.

THEOREM 2. *The set* (3) *of all rigid motions of the plane forms a group.*

The set of all translations of the plane, including the identity, is represented by the equations

$$(4) \qquad x' = x + a, \qquad y' = y + b,$$

where a and b are arbitrary constants. Let us prove analytically that this set forms a group. Two arbitrarily chosen, but fixed, transformations of the set are

$$\begin{aligned} \bar{x} &= x + a_1, \qquad & x' &= \bar{x} + a_2, \\ \bar{y} &= y + b_1, \qquad & y' &= \bar{y} + b_2, \end{aligned}$$

where a_1, b_1, a_2, b_2 are fixed. Their product

$$x' = x + a_1 + a_2, \qquad y' = y + b_1 + b_2,$$

belongs to the set, being the particular transformation for which $a = a_1 + a_2$, $b = b_1 + b_2$. Moreover, if an arbitrary but fixed transformation of the set, say

$$x' = x + a_0, \qquad y' = y + b_0,$$

is chosen, its inverse

$$x = x' - a_0, \qquad y = y' - b_0,$$

clearly belongs to the set.

THEOREM 3. *The set* (4) *of all translations of the plane forms a group.*

THEOREM 4. *All the rotations about a given point, for example, the origin, form a group.*

The proof of the latter theorem we leave to the reader.

The group of translations is a subgroup of the group of rigid motions; the translations (4) are precisely those rigid motions (3) for which $\theta = 0$. Similarly, the group of rotations about the origin is a subgroup of the group of rigid motions.

The equations (3) of the group of rigid motions contain three arbitrary constants, or parameters, a, b, and θ, to which values can be given independently. We express this fact by saying that the group depends on three parameters or is a *three-parameter group.*

EXERCISES

1. Verify the statement that the transformations (2) form a group.

2. Prove analytically that the rotations about the origin form a group.

3. On how many parameters does the group of translations depend? On how many does the group of rotations about a given point depend?

5. Invariants. We know that the general rigid motion,

(1) $$x' = x \cos \theta - y \sin \theta + a, \qquad y' = x \sin \theta + y \cos \theta + b,$$

preserves distance. Let us verify this fact analytically.

Let $P_1 : (x_1, y_1)$ and $P_2 : (x_2, y_2)$ be two arbitrary but fixed points, and let $P_1' : (x_1', y_1')$ and $P_2' : (x_2', y_2')$ be respectively the points into which they are carried by (1). Then

$$x_1' = x_1 \cos \theta - y_1 \sin \theta + a, \qquad y_1' = x_1 \sin \theta + y_1 \cos \theta + b,$$
$$x_2' = x_2 \cos \theta - y_2 \sin \theta + a, \qquad y_2' = x_2 \sin \theta + y_2 \cos \theta + b.$$

Hence,

$$x_2' - x_1' = (x_2 - x_1) \cos \theta - (y_2 - y_1) \sin \theta,$$
$$y_2' - y_1' = (x_2 - x_1) \sin \theta + (y_2 - y_1) \cos \theta.$$

Squaring and adding, we have

$$(x_2' - x_1')^2 + (y_2' - y_1')^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2.$$

Hence $P_1'P_2' = P_1P_2$ and the verification is complete.

The expression

(2) $$(x_2 - x_1)^2 + (y_2 - y_1)^2$$

in the coordinates of the given points is always equal to the *same* expression in the coordinates of the transformed points. To express this fact, we say:

The expression (2) *is an invariant of the points* (x_1, y_1), (x_2, y_2) *with respect to the group of rigid motions.*

This is merely the analytic counterpart of the statement that distance is preserved by every rigid motion.

Another important property which is preserved by all rigid motions is the angle between two lines. Let the lines, arbitrarily chosen but fixed, be

$$(3) \qquad a_1x + a_2y + a_3 = 0, \qquad b_1x + b_2y + b_3 = 0.$$

To find the lines into which they are carried by the general rigid motion (1), we need the inverse of (1). This does not mean that we have to solve equations (1) for x, y; for, since the inverse is a rigid motion, its equations are necessarily of the form (1) and so can be written as

$$(4) \quad x = x' \cos\phi - y' \sin\phi + c, \qquad y = x' \sin\phi + y' \cos\phi + d,$$

where c, d, ϕ are parameters which are related to the original parameters a, b, θ of (1) in a way which is perfectly definite but, for our present purpose, immaterial.

Substituting for x and y in (3) the values given by (4), we obtain, as the equations of the transformed lines,

$$(5) \qquad a_1'x' + a_2'y' + a_3' = 0, \qquad b_1'x' + b_2'y' + b_3' = 0,$$

where

$$(6) \quad \begin{aligned} a_1' &= a_1\cos\phi + a_2\sin\phi, & b_1' &= b_1\cos\phi + b_2\sin\phi,\\ a_2' &= -a_1\sin\phi + a_2\cos\phi, & b_2' &= -b_1\sin\phi + b_2\cos\phi,\\ a_3' &= a_1c + a_2d + a_3, & b_3' &= b_1c + b_2d + b_3. \end{aligned}$$

To show that the angle from the first of the lines (3) to the second is equal to the corresponding angle for the lines (5), it suffices to show that

$$\frac{a_1'b_2' - a_2'b_1'}{a_1'b_1' + a_2'b_2'} = \frac{a_1b_2 - a_2b_1}{a_1b_1 + a_2b_2}.$$

The truth of this equality, regardless of the values of c, d, and ϕ, is readily established by means of the relations (6). Hence

The expression for the tangent of the angle from the first of the lines (3) *to the second, namely*

$$(7) \qquad \frac{a_1b_2 - a_2b_1}{a_1b_1 + a_2b_2},$$

*is an invariant of the two lines with respect to the group of rigid motions.**

* Cayley (1821–1895) and Sylvester (1819–1897), two English mathematicians, deserve the major credit for the creation of the theory of invariants.

EXERCISES

1. Check in detail every step of the foregoing proof that (7) is an invariant.

2. Show that

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

is an invariant of the three points (x_1, y_1), (x_2, y_2), (x_3, y_3) with respect to the group of rigid motions. What is the geometrical meaning of this invariant?

3. Prove that the slope of a line is an invariant with respect to the group of translations.

4. Establish $x_1y_2 - x_2y_1$ as an invariant of the two points (x_1, y_1), (x_2, y_2) with respect to the group of rotations about the origin. What is its geometrical significance?

5. Show that

$$\frac{a_1x_0 + a_2y_0 + a_3}{\sqrt{a_1^2 + a_2^2}}$$

is an invariant of the point (x_0, y_0) and the line $a_1x + a_2y + a_3 = 0$ with respect to the group of rigid motions.

6. Prove that

$$x_0^2 + y_0^2 + a_1x_0 + a_2y_0 + a_3$$

is an invariant of the point (x_0, y_0) and the circle

$$x^2 + y^2 + a_1x + a_2y + a_3 = 0$$

with respect to the group of rigid motions.

B. One-Dimensional Projective Transformations

6. Projections and Projective Correspondences. A projection of a line L on a line L' we have found to possess the two following properties.

Theorem 1. *A projection of a line L upon a line L' establishes between the ranges of points on L and L' a one-to-one correspondence and preserves cross ratio.*

Is the converse true? Can every one-to-one correspondence between two ranges of points which preserves cross ratio be established by a projection?

In showing that this question is to be answered affirmatively, we first prove the following theorem.

Theorem 2. *There exists a unique correspondence between two ranges of points which is one-to-one and preserves cross ratio, and which orders to three given distinct points A, B, C of the one range three prescribed distinct points A', B', C' of the other range.*

If a correspondence of the desired type, ordering A' to A, B' to B, and C' to C, exists, it must order to an arbitrarily chosen point P of the one range a point P' of the other range such that the cross ratio $(A'B', C'P')$ equals the known cross ratio $(A\, B, C\, P)$:

$$(1) \qquad (A'B', C'P') = (AB, CP).$$

By Ch. VI, § 7, Th. 1, there is only one point P' with this property. Consequently, there is at most one correspondence of the desired type, that defined by (1).

The correspondence defined by (1) is one-to-one and orders A' to A, B' to B, and C' to C. It remains to show that it always preserves cross ratio, that is, that, if $P_1 \longleftrightarrow P'_1$, $P_2 \longleftrightarrow P'_2$, $P_3 \longleftrightarrow P'_3$, $P_4 \longleftrightarrow P'_4$ are *any* four distinct pairs of corresponding points, then

$$(2) \qquad (P'_1P'_2, P'_3P'_4) = (P_1P_2, P_3P_4).$$

According to Ch. III, § 6, Ex. 7, we can choose homogeneous coordinates a and b of A and B so that C has the coordinates $a + b$. Similarly, we can take the coordinates a' and b' of A' and B' so that C' is the point $a' + b'$. If, then, P has the coordinates $a + \mu\, b$, P' must have, by (1), the coordinates $a' + \mu\, b'$. In particular, if P_1, P_2, P_3, P_4 have the coordinates

$$a + \mu_1 b, \qquad a + \mu_2 b, \qquad a + \mu_3 b, \qquad a + \mu_4 b,$$

P'_1, P'_2, P'_3, P'_4 have the coordinates

$$a' + \mu_1 b', \qquad a' + \mu_2 b', \qquad a' + \mu_3 b', \qquad a' + \mu_4 b'.$$

It follows that the cross ratios in (2) are equal, for the values of the cross ratio depend only on the μ's and these are the same in both cases.

We are now ready to prove the converse of Theorem 1. Let a correspondence between the ranges of points on the lines L and L' be given which is one-to-one and preserves cross ratio, and let $A \leftrightarrow A'$, $B \leftrightarrow B'$, $C \leftrightarrow C'$ be three distinct pairs of corresponding points. Move L' rigidly so that A' coincides with A and then draw BB' and CC' intersecting in O, as shown in Fig. 8. The projection from O of L on L' establishes a correspondence which is one-to-one, preserves cross ratio, and orders A' to A, B' to B, and C' to C. But these are precisely the properties of the given corre-

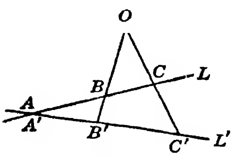


FIG. 8

spondence, and since, by Th. 2, there can be but one correspondence with these properties, the given correspondence is identical with that established by the projection.

Theorem 1 and its converse may be summarized as follows.

THEOREM 3. *The correspondences established by projections of one line upon another are identical with the correspondences between two ranges of points which are one-to-one and preserve cross ratio.*

The theorem serves two purposes. In the first place, it portrays the inherent simplicity of the correspondences established by projections. Secondly, it enables us to replace the study of projections by that of the correspondences with the two properties described; in other words, it enables us to discuss the effects of projection without being hampered by the details of the process itself.

We shall, then, suppress the process of projection and devote our attention wholly to the correspondences effected thereby. These correspondences it is natural to call projective.

DEFINITION. *A correspondence between two ranges of points which is one-to-one and preserves cross ratio is known as a one-dimensional projective correspondence.*

Theorem 2 now becomes

THEOREM 4. *A projective correspondence between two ranges of points is uniquely determined by three distinct pairs of corresponding points.*

Correspondences and Transformations. In a correspondence between two ranges, R and R', two corresponding points, P and P', are reciprocally related: to each corresponds the other. For this reason, we use the double arrow, $P \leftrightarrow P'$, $R \leftrightarrow R'$, to indicate the correspondence.

A correspondence, $R \leftrightarrow R'$, determines two transformations, the transformation $R \rightarrow R'$, which transforms the points of R into those of R', and the inverse transformation, $R' \rightarrow R$, which carries the points of R' into those of R.

The transformations determined by projective correspondences we shall call *projective transformations.*

7. One-Dimensional Projective Transformations. We proceed to deduce the equations of an arbitrary projective correspondence and the resulting projective transformations.

Let the projective correspondence, between the ranges of points on

the lines L and L', be determined by the three distinct pairs of corresponding points, $P_1 \leftrightarrow P'_1$, $P_2 \leftrightarrow P'_2$, $P_3 \leftrightarrow P'_3$. It is then defined by the equation

(1) $$(P'_1P'_2, P'_3P') = (P_1P_2, P_3P),$$

where $P \leftrightarrow P'$ is an arbitrary pair of corresponding points.

Introduce on L and L' arbitrarily chosen systems of Cartesian coordinates; see Ch. IX, § 1. Let x_1, x_2, x_3, and x be the coordinates of P_1, P_2, P_3, and P, and x'_1, x'_2, x'_3, and x', those of P'_1, P'_2, P'_3, and P'. Equation (1) of the projective correspondence then becomes

(2) $$\frac{(x'_3 - x'_1)(x' - x'_2)}{(x'_3 - x'_2)(x' - x'_1)} = \frac{(x_3 - x_1)(x - x_2)}{(x_3 - x_2)(x - x_1)}.$$

To find the equation of the related transformation, say, of the range on L into that on L', we have merely to solve (2) for x' in terms of x. The work will be facilitated, if we set

$$\frac{x'_3 - x'_1}{x'_3 - x'_2} = c', \qquad \frac{x_3 - x_1}{x_3 - x_2} = c.$$

Since x_1, x_2, x_3, x'_1, x'_2, x'_3 are constants, c and c' are constants. Equation (2) now becomes

$$c'\frac{x' - x'_2}{x' - x'_1} = c\frac{x - x_2}{x - x_1}.$$

Solving for x', we obtain the desired equation:

(3) $$x' = \frac{a_1x + a_2}{b_1x + b_2},$$

where a_1, a_2, b_1, b_2 are constants; for example, $b_1 = c - c'$.

The transformation (3) takes on a more symmetric form on the introduction of *homogeneous* coordinates on L and L'. Setting

$$x = \frac{x_1}{x_2}, \qquad x' = \frac{x'_1}{x'_2},$$

we have

(4) $$\frac{x'_1}{x'_2} = \frac{a_1x_1 + a_2x_2}{b_1x_1 + b_2x_2}.$$

This equation says that x'_1, x'_2 are proportional to $a_1x_1 + a_2x_2$,

$b_1x_1 + b_2x_2$. Consequently

$$\rho x_1' = a_1x_1 + a_2x_2, \qquad \rho x_2' = b_1x_1 + b_2x_2, \qquad \rho \neq 0. \tag{5}$$

Conversely, we can pass from (5) through (4) back to (3). Hence equations (5) and equation (3) represent the same transformation.

The right-hand sides of equations (5) are linear in x_1, x_2. Accordingly, we call a transformation of the form (5) in homogeneous coordinates, or a transformation of the form (3) in nonhomogeneous coordinates, a one-dimensional *linear transformation.*

We may now state our result as follows.

THEOREM. *Every projective transformation of a range of points into a range of points is a linear transformation.*

EXERCISES

1. Find the equation of the projective transformation which carries the points 0, 1, 2 of L respectively into the points -1, 0, -2 of L'. Write the equations of the transformation in homogeneous coordinates and find the equations of the inverse transformation in both kinds of coordinates. Determine the vanishing point on each line.

2. Find the equation of the projective transformation which transforms the points 2, 4, and the point at infinity on L respectively into the points -1, 1, and the point at infinity on L'.

8. One-Dimensional Linear Transformations.

Let

$$x' = \frac{a_1 x + a_2}{b_1 x + b_2}, \quad \text{or} \quad \rho x_1' = a_1x_1 + a_2x_2, \quad \rho x_2' = b_1x_1 + b_2x_2, \qquad \rho \neq 0, \tag{1}$$

be an arbitrary linear transformation of the points of a line L into those of a line L'.

The formal process of finding the inverse transformation, applied to the right-hand equations in (1), yields the equations

$$\Delta x_1 = \rho(\ \ b_2 x_1' - a_2 x_2'), \qquad \Delta x_2 = \rho(-b_1 x_1' + a_1 x_2'), \tag{2}$$

where

$$\Delta = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}.$$

The determinant Δ is known as the *determinant of the linear transformation* (1). If $\Delta = 0$, equations (2) fail to define the inverse

transformation.* We exclude this case and assume henceforth that $\Delta \neq 0$.

Equations (2) then yield, as the inverse of (1), the linear transformation

$$(3) \qquad x = \frac{b_2 x' - a_2}{-b_1 x' + a_1}, \qquad \text{or} \qquad \begin{aligned} \sigma x_1 &= b_2 x_1' - a_2 x_2', \\ \sigma x_2 &= -b_1 x_1' + a_1 x_2', \end{aligned} \qquad \sigma \neq 0.$$

It is to be noted that the determinant of the inverse is equal to that of the given transformation.

Theorem 1. *Every linear transformation of a range of points into a range of points is a projective transformation.*

This is the converse of the Theorem of § 7. To prove it, we have to show that the transformation (1) has the two properties characteristic of a projective transformation. It has the first of these properties: it establishes a one-to-one correspondence between the points of L and L'. For, (1) orders to each point of L a single point of L' and the inverse transformation (3) orders to each point of L' a single point of L.

It also has the second property: it preserves cross ratio. Let (x_1, x_2), (y_1, y_2), (z_1, z_2), (t_1, t_2) be four distinct points on L and let (x_1', x_2'), (y_1', y_2'), (z_1', z_2'), (t_1', t_2') be the four points on L' into which they are carried. Then, for example,

$$(4) \qquad \begin{aligned} \rho_1 x_1' &= a_1 x_1 + a_2 x_2, \\ \rho_1 x_2' &= b_1 x_1 + b_2 x_2, \end{aligned} \qquad \begin{aligned} \rho_3 z_1' &= a_1 z_1 + a_2 z_2, \\ \rho_3 z_2' &= b_1 z_1 + b_2 z_2. \end{aligned}$$

It is to be emphasized that the constant of proportionality, ρ, may vary from one pair of points to another, inasmuch as the homogeneous coordinates of a point admit, themselves, a constant of proportionality. We denote the values of ρ for the four pairs of points by ρ_1, ρ_2, ρ_3, ρ_4.

* When $\Delta = 0$, a_1, a_2 and b_1, b_2 are proportional. Hence $a_1 x + a_2$ and $b_1 x + b_2$ are proportional. If we exclude the case in which a_1, a_2, b_1, b_2 are all zero, this means that the transformation carries every point of L, save one, into the same point of L'. For example, the transformation,

$$x' = \frac{3x - 3}{2x - 2} = \frac{3(x - 1)}{2(x - 1)},$$

carries every point of L, other than $x = 1$, into the point $x' = 3/2$ of L'. We have here the analytic analog of a projection of L on L' from a center which lies on L'. We have excluded this type of projection, and accordingly rule out the corresponding type of linear transformation.

We desire to show that the cross ratio of the four points on L in the order given is equal to the corresponding cross ratio of the transformed four points:

$$(5) \qquad \frac{|z'\,x'|\cdot|t'\,y'|}{|z'\,y'|\cdot|t'\,x'|}=\frac{|z\,x|\cdot|t\,y|}{|z\,y|\cdot|t\,x|}\,.$$

By virtue of (4), the determinant $|z'\,x'|$ becomes

$$\begin{vmatrix} z_1' & x_1' \\ z_2' & x_2' \end{vmatrix} = \frac{1}{\rho_1\rho_3}\begin{vmatrix} a_1z_1+a_2z_2 & a_1x_1+a_2x_2 \\ b_1z_1+b_2z_2 & b_1x_1+b_2x_2 \end{vmatrix} = \frac{\Delta}{\rho_1\rho_3}\begin{vmatrix} z_1 & x_1 \\ z_2 & x_2 \end{vmatrix},$$

for the intermediate determinant is the product of the determinants Δ and $|z\,x|$; see Ch. 1, § 6, Ex. 5. Thus, each determinant on the left-hand side of (5) can be replaced by a certain multiple of the corresponding determinant on the right-hand side. The left-hand side then reduces to the right-hand side, and our theorem is proved.

We have now achieved a major goal. The projective transformations of the points of one range into the points of a second range, which hitherto have been defined only geometrically, are now characterized analytically. Theorem 1 and the Theorem of § 7 tell us that they are identical with the linear transformations of the points of the one range into those of the other range.

The Group of Projective or Linear Transformations. Suppose that we think of L and L' as the same line and of x and x' as referred to the same coordinate system on this line. Equation (1) then represents the general projective, or linear, transformation of the line into itself.

The totality of these transformations forms a group; for, the inverse of a linear transformation we have already seen to be linear, and the product of two linear transformations can readily be proved to be linear (Ex. 2).

Theorem 2. *The set of all projective transformations of a range of points into itself forms a group.*

In establishing equation (5), we proved that

$$\frac{|z\,x|\cdot|t\,y|}{|z\,y|\cdot|t\,x|}$$

is an invariant of the four points (x_1, x_2), (y_1, y_2), (z_1, z_2), (t_1, t_2) with respect to the group of linear transformations.

EXERCISES

1. Prove, by means of nonhomogeneous coordinates, that a linear transformation preserves cross ratio.

2. Show that the product of two linear transformations is a linear transformation. Note that it is necessary to show that the determinant of the product transformation does not vanish. This is done by proving that this determinant is equal to the product of the determinants of the given transformations.

3. Show that the translations $x' = x + b$ are the only rigid motions of a line into itself and that they form a group which is a subgroup of the group of projective transformations of the line into itself.

Projective Transformations of Pencils of Lines

Definition. *A transformation of the lines of a pencil into the lines of a second pencil which establishes a one-to-one correspondence between the lines of the two pencils and preserves cross ratio is a projective transformation.*

4. Show that there is a unique projective transformation of the lines of one pencil into those of a second pencil which carries three given distinct lines of the first pencil respectively into three prescribed distinct lines of the second.

5. Prove that a projective transformation of one pencil of lines into a second is a linear transformation, that is, a transformation of the form

$$u' = \frac{a_1 u + a_2}{b_1 u + b_2} \quad \text{or} \quad \begin{aligned} \rho u_1' &= a_1 u_1 + a_2 u_2, \\ \rho u_2' &= b_1 u_1 + b_2 u_2, \end{aligned} \qquad \Delta \neq 0,$$

where u and u' are nonhomogeneous, and (u_1, u_2) and (u_1', u_2') homogeneous, coordinates in the two pencils; see Ch. IX, § 1.

6. Show that a linear transformation of the lines of one pencil into those of a second is a projective transformation.

C. Two-Dimensional Projective Transformations

9. Two-Dimensional Linear Transformations. In Part B we analyzed the projections of a line upon a line and characterized them geometrically and analytically. We propose to do the same for the projections of a plane upon a plane and begin by stating the properties of these projections with which we are already familiar.

Theorem 1. *A projection of a plane upon a plane establishes a one-to-one correspondence between the points of the two planes, and a one-to-one correspondence between the lines of the two planes, and preserves cross ratio.*

The converse is true: every transformation of the one plane into the other which has the three properties enumerated can be established

by projections.* Projections are thus characterized geometrically as being, in effect, transformations which possess these three properties. Henceforth, we suppress the actual process of projection and concentrate our attention on the transformations in question. These we call two-dimensional projective transformations.

DEFINITION. *A transformation of the points of a plane M into the points of a plane M' which (a) establishes a one-to-one correspondence between the points of M and M', (b) establishes a one-to-one correspondence between the lines of M and M', and (c) preserves cross ratio, is known as a two-dimensional projective transformation.*

In the one-dimensional case, the projective transformations were identical with the linear transformations. This is also true in the present case. To prove it, we begin by developing the theory of two-dimensional linear transformations.

Linear Transformations. The general linear transformation of the points of M into the points of M' is, in homogeneous coordinates,

$$\begin{aligned} \rho x_1' &= a_{11}x_1 + a_{12}x_2 + a_{13}x_3, \\ \rho x_2' &= a_{21}x_1 + a_{22}x_2 + a_{23}x_3, \\ \rho x_3' &= a_{31}x_1 + a_{32}x_2 + a_{33}x_3, \end{aligned} \qquad \rho \neq 0. \tag{1}$$

One advantage of the double-subscript notation which we have here introduced is that it permits us to write the equations of the transformation in the condensed form

$$\rho x_1' = \sum_{j=1}^{3} a_{1j}x_j, \qquad \rho x_2' = \sum_{j=1}^{3} a_{2j}x_j, \qquad \rho x_3' = \sum_{j=1}^{3} a_{3j}x_j,$$

or, more briefly,

$$\rho x_i' = \sum_{j=1}^{3} a_{ij}x_j, \qquad (i = 1, 2, 3). \tag{1'}$$

The determinant of the coefficients in (1), commonly denoted by $|a_{ij}|$, is known as the *determinant of the transformation.* If $|a_{ij}| = 0$, it can be shown that (1) fails to establish a one-to-one correspondence between the points of M and M'. Accordingly, we exclude this case and assume that $|a_{ij}| \neq 0$.

* For the proof of the converse, the reader is referred to an article in the *American Mathematical Monthly*, Vol. XXXV (1928), pp. 412–415. He will do well, however, to postpone consulting this article until after he has read Ch. XI.

Equations (1) can then be solved by Cramer's rule for x_1, x_2, x_3. The result is the inverse transformation of (1):

$$\begin{aligned} \sigma x_1 &= A_{11}x_1' + A_{21}x_2' + A_{31}x_3', \\ \sigma x_2 &= A_{12}x_1' + A_{22}x_2' + A_{32}x_3', \\ \sigma x_3 &= A_{13}x_1' + A_{23}x_2' + A_{33}x_3', \end{aligned} \qquad \sigma \neq 0, \tag{2}$$

or

$$\sigma x_i = \sum_{j=1}^{3} A_{ji}x_j', \qquad (i = 1, 2, 3), \tag{2'}$$

where A_{ij} is the cofactor of a_{ij} in the determinant $|a_{ij}|$.* According to Ch. I, § 6, Ex. 1, the determinant $|A_{ij}|$ of the inverse transformation has the value $|a_{ij}|^2$ and hence is not zero.

The transformation (1) establishes a one-to-one correspondence between the points of M and M'; for, (1) orders to each point of M a unique point of M', and (2) orders to each point of M' a unique point of M.

The transformation (1) establishes a one-to-one correspondence between the lines of M and M'. In verifying this statement, we shall use the following lemma, which is readily proved.

Lemma. *If the transformation* (1) *carries the points y and z in the plane M into the points y' and z' in the plane M', it carries the point $y + \mu z$ in M into the point $\rho_1 y' + \mu\rho_2 z'$ in M', where ρ_1 and ρ_2 are, respectively, the values of the constant of proportionality ρ for the pairs of points $y \to y'$ and $z \to z'$.*

If the points y and z in M are distinct, the points y' and z' in M' are also distinct, for otherwise the correspondence between the points of M and M' would not be one-to-one. Consequently, if we choose arbitrarily in M a line L and think of y and z as two distinct points of L, y' and z' will determine in M' a line L'. As μ varies, the point $y + \mu z$ traces the line L; then the transformed point $\rho_1 y' + \mu\rho_2 z'$ traces L'. Thus, to the line L in M corresponds the unique line L' in M'. Similarly, using the inverse of the transformation, we can prove that to a given line in M' corresponds a unique line in M.

The transformation (1) preserves cross ratio. It suffices to establish this fact in the case of points. Let

$$y, \qquad z, \qquad y + \mu_1 z, \qquad y + \mu_2 z$$

* In (1), a_{ij} is the coefficient of x_j in the expression for x_i'; in (2), A_{ij} is the coefficient of x_i' in the expression for x_j. Thus, the first subscript of an a or an A pertains always to an x', whereas the second pertains always to an x.

be four distinct collinear points in M. The points in M' into which they are carried have, according to the Lemma, the coordinates

$$y', \qquad z', \qquad \rho_1 y' + \mu_1 \rho_2 z', \qquad \rho_1 y' + \mu_2 \rho_2 z'.$$

The cross ratio of each set of points, in the order given, is μ_1/μ_2. Hence cross ratio is preserved.

We have shown that the transformation (1) has the three properties characteristic of a projective transformation. Hence:

THEOREM 2. *Every linear transformation of the points of M into the points of M' is a projective transformation.*

The theorem constitutes half the proof that the two-dimensional projective and linear transformations are identical.

EXERCISES

1. Find the equations of the inverse of the linear transformation

$$\rho x_1' = 2x_1 - x_2 + x_3, \qquad \rho x_2' = x_1 + 2x_2 - x_3, \qquad \rho x_3' = x_1 + x_2 + x_3.$$

Write the equations in nonhomogeneous coordinates of both the given transformation and the inverse.

2. Into what line is the line $3x_1 + x_2 - 2x_3 = 0$ carried by the transformation of Ex. 1?

3. What is the equation of the vanishing line in each plane, in the case of the transformation of Ex. 1? In the case of the transformation (1) of the text?

4. Prove the Lemma of the text.

5. Show that the product of two linear transformations is a linear transformation; see § 8, Ex. 2.

10. The Fundamental Theorem.

The next step in our development is to establish the following theorem.

THEOREM 1. *There exists a unique linear transformation of the points of a plane M into the points of a plane M' which carries four given points, no three collinear, in M respectively into four prescribed points, no three collinear, in M'.**

Special Case A. We shall prove the theorem first in the case when the given points in M are the specially chosen points (1, 0, 0), (0, 1, 0), (0, 0, 1), (1, 1, 1), and the prescribed points a', b', c', d' in M' are subject only to the restriction that no three be collinear.

* This theorem is due to Moebius, who was the first to discuss projective transformations systematically.

Suppose that there exists a linear transformation

$$(1)\qquad \begin{aligned} \rho x_1' &= a_{11}x_1 + a_{12}x_2 + a_{13}x_3, \\ \rho x_2' &= a_{21}x_1 + a_{22}x_2 + a_{23}x_3, \\ \rho x_3' &= a_{31}x_1 + a_{32}x_2 + a_{33}x_3, \end{aligned} \qquad |a_{ij}| \neq 0,$$

which carries (1, 0, 0) into (a_1', a_2', a_3'), (0, 1, 0) into b', (0, 0, 1) into c', and (1, 1, 1) into d'. Then

$$(2)\qquad \begin{array}{llll} \rho_1 a_1' = a_{11}, & \rho_2 b_1' = a_{12}, & \rho_3 c_1' = a_{13}, & -\rho_4 d_1' = \sum_{j=1}^{3} a_{1j}, \\ \rho_1 a_2' = a_{21}, & \rho_2 b_2' = a_{22}, & \rho_3 c_2' = a_{23}, & -\rho_4 d_2' = \sum_{j=1}^{3} a_{2j}, \\ \rho_1 a_3' = a_{31}, & \rho_2 b_3' = a_{32}, & \rho_3 c_3' = a_{33}, & -\rho_4 d_3' = \sum_{j=1}^{3} a_{3j}. \end{array}$$

Twelve linear homogeneous equations, to be solved for thirteen unknowns: the nine a_{ij}'s and the four ρ's. A first step toward the solution consists in eliminating the nine a_{ij}'s by substituting the values given for them by the first nine equations into the last three. There results the following system of three homogeneous equations in the four ρ's:

$$\begin{aligned} a_1'\rho_1 + b_1'\rho_2 + c_1'\rho_3 + d_1'\rho_4 &= 0, \\ a_2'\rho_1 + b_2'\rho_2 + c_2'\rho_3 + d_2'\rho_4 &= 0, \\ a_3'\rho_1 + b_3'\rho_2 + c_3'\rho_3 + d_3'\rho_4 &= 0. \end{aligned}$$

Since no three of the four points a', b', c', d' are collinear, the rank of the system is three. Hence, an arbitrary solution, other than 0, 0, 0, 0, is

$$(3\,a)\qquad \rho_1 = k|b'\ c'\ d'|, \qquad \rho_2 = -k|a'\ c'\ d'|, \qquad \rho_3 = k|a'\ b'\ d'|, \qquad \rho_4 = -k|a'\ b'\ c'|,$$

where k is an arbitrary constant, not zero. Substituting these values of the ρ's into the first nine equations in (2), we find as the values of the a_{ij}'s:

$$(3\,b)\qquad \begin{array}{lll} a_{11} = k|b'\ c'\ d'|a_1', & a_{12} = -k|a'\ c'\ d'|b_1', & a_{13} = k|a'\ b'\ d'|c_1', \\ a_{21} = k|b'\ c'\ d'|a_2', & a_{22} = -k|a'\ c'\ d'|b_2', & a_{23} = k|a'\ b'\ d'|c_2', \\ a_{31} = k|b'\ c'\ d'|a_3', & a_{32} = -k|a'\ c'\ d'|b_3', & a_{33} = k|a'\ b'\ d'|c_3'. \end{array}$$

Formulas (3 a) and (3 b) give all the solutions of the thirteen equations (2), except the solution consisting exclusively of zeros. Consequently, if there is a linear transformation (1) with the desired property, its coefficients are given by (3 b). Moreover, since two sets

of a_{ij}'s which correspond to two values of k in (3 b) are proportional, this linear transformation is unique.

The linear transformation does carry the given points in M into the specified points in M', provided no one of the ρ's, as given by (3 a), is zero. But no ρ can vanish, inasmuch as no three of the four points a', b', c', d' are collinear.

Of one more fact we must make sure, namely, that $|a_{ij}| \neq 0$. The value of $|a_{ij}|$, obtained from (2) or (3 b), is

$$|a_{ij}| = \rho_1\rho_2\rho_3|a'\,b'\,c'|.$$

Hence $|a_{ij}| \neq 0$ and the proof is complete.

Special Case B. It follows that there exists a unique linear transformation carrying four arbitrary points a, b, c, d, no three collinear, in M respectively into the points (1, 0, 0), (0, 1, 0), (0, 0, 1), (1, 1, 1) in M'. This transformation is essentially the inverse of the one just determined.

General Case. Let the four given points in M be a, b, c, d and the corresponding prescribed points in M', a', b', c', d'. Insert a new plane $\overline{M}$ and mark in it the four points (1, 0, 0), (0, 1, 0), (0, 0, 1), (1, 1, 1). According to the special cases, there exists a unique linear transformation T_1 of the points of M into those of $\overline{M}$ which carries a into (1, 0, 0), b into (0, 1, 0), c into (0, 0, 1), and d into (1, 1, 1); and, a unique linear transformation T_2 of the points of $\overline{M}$ into those of M' which carries (1, 0, 0) into a', (0, 1, 0) into b', (0, 0, 1) into c', and (1, 1, 1) into d'. The product, $T = T_1T_2$, of these transformations is a linear transformation (§ 9, Ex. 5) of the points of M into those of M' which carries a into a', b into b', c into c', and d into d'.

It remains to show that T is the only linear transformation which satisfies the requirements. Suppose that there is a second, S. The product ST_2^{-1} is a linear transformation of M into $\overline{M}$ which carries a, b, c, d respectively into (1, 0, 0), (0, 1, 0), (0, 0, 1), (1, 1, 1). But there is only one linear transformation doing this, namely T_1. Consequently, $ST_2^{-1} = T_1$, and therefore $ST_2^{-1}T_2 = T_1T_2$. But $T_2^{-1}T_2$ is the identity, and $T_1T_2 = T$. Hence, $S = T$.*

* The accompanying diagrams help to clarify the deduction of $S = T_1T_2$ from $ST_2^{-1} = T_1$. The first diagram depicts the hypothesis: $ST_2^{-1} = T_1$. The second may be thought of as obtained from the first by replacing T_2^{-1} by T_2 and reversing the corresponding arrow. It tells us that $S = T_1T_2$ and hence that $S = T$.

$$\begin{tikzcd} {} \arrow[rr, "S"] \arrow[dr, "T_1"'] & & {} \arrow[dl, "T_2^{-1}"] \\ & {} & \end{tikzcd} \qquad \begin{tikzcd} {} \arrow[rr, "S"] \arrow[dr, "T_1"'] & & {} \\ & {} \arrow[ur, "T_2"'] & \end{tikzcd}$$

EXERCISES

1. Find the equations of the linear transformation which carries $(1, 0, 1)$, $(0, 1, 1)$, $(1, 1, 1)$, $(0, 0, 1)$ into $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$, $(1, 1, 1)$ respectively.

2. Prove analytically that there is a unique linear transformation of the points of a line L into those of a line L' which carries three given distinct points of L respectively into three given distinct points of L'.

11. Two-Dimensional Projective Transformations. We are now in a position to complete the proof that the two-dimensional linear and projective transformations are identical. We have already shown, in § 9, that every linear transformation is projective. It remains to show that every projective transformation is a linear transformation.

Let T be a projective transformation of the points of a plane M into those of a plane M'. Choose four points A_1, A_2, A_3, D, no three collinear, in M, and let the points in M' into which they are carried by T be respectively A_1', A_2', A_3', D'. These points also have the property, by part (*b*) of the definition of a projective transformation (§ 9), that no three lie on a line. There exists, therefore, a unique linear transformation, S, of the points of M into those of M' which carries A_1 into A_1', A_2 into A_2', A_3 into A_3', and D into D'.

To prove that T is the same as S, we shall show that they both carry an arbitrarily chosen point of M into the same point in M'.

We know already that T and S both carry A_1, A_2, A_3, D into A_1', A_2', A_3', D'. Moreover, since every linear transformation is projective, S is projective as well as T. Hence both carry, for example, a_3 into a_3', d_3 into d_3', and therefore D_3 into D_3' (Fig. 9).

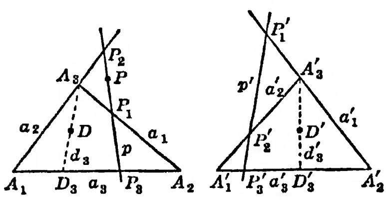


FIG. 9

Since both S and T order to the points A_1, A_2, D_3 on a_3 the points A_1', A_2', D_3' on a_3', they both transform an arbitrarily chosen point on a_3 into the same point on a_3'. For, they both establish between the ranges of points on a_3 and a_3' projective correspondences (Ex. 1), and a projective correspondence between two ranges of points is uniquely determined by three distinct pairs of corresponding points.

We have thus proved that the two transformations carry the points on the sides of the triangle $A_1A_2A_3$ into the same points on the sides of the triangle $A_1'A_2'A_3'$.

Suppose, finally, that P is an arbitrarily chosen point, not on a side

of $A_1A_2A_3$. Let p be a line through P which intersects the sides of $A_1A_2A_3$ in the three distinct points P_1, P_2, P_3. Since S and T carry P_1, P_2, P_3 into the same points P_1', P_2', P_3', both carry p into the same line p', and transform an arbitrarily chosen point on p into the same point on p'. In particular, they both order to P the same point P'.

THEOREM. *The projective transformations of the points of a plane M into the points of a plane M' are identical with the linear transformations of the points of M into the points of M'.*

It follows that the fundamental theorem for linear transformations is valid also for projective transformations.

EXERCISE 1. Prove that, if a projective transformation of the points of a plane M into those of a plane M' carries the line L in M into the line L' in M', it establishes between the ranges of points on L and L' a projective correspondence.

12. Linear and Projective Transformations of Planes of Lines. Thus far we have considered projective transformations from the point of view of point geometry. But a projection of a plane M upon a plane M' can be equally well considered from the point of view of line geometry. It establishes a transformation of the lines of M into the lines of M'; to each line L of M is ordered the line L' of M' in which the plane determined by L and the center of projection intersects M'.

It devolves on us to reconsider the material of the preceding paragraphs from the point of view of line geometry. This we do in brief, leaving the proofs of the theorems to the reader.

DEFINITION. *A transformation of the lines of a plane M into the lines of a plane M' which (a) establishes a one-to-one correspondence between the lines of M and M', (b) establishes a one-to-one correspondence between the points of M and M', and (c) preserves cross ratio, is called a projective transformation.*

The general linear transformation of the lines u of the plane M into the lines u' of the plane M' is

$$(1)\qquad \begin{aligned} \rho u_1' &= b_{11}u_1 + b_{12}u_2 + b_{13}u_3, \\ \rho u_2' &= b_{21}u_1 + b_{22}u_2 + b_{23}u_3, \\ \rho u_3' &= b_{31}u_1 + b_{32}u_2 + b_{33}u_3, \end{aligned} \qquad |b_{ij}| \neq 0,$$

or

$$\rho u_i' = \sum_{j=1}^{3} b_{ij}u_j. \qquad (i = 1, 2, 3).$$

Its inverse is

$$\sigma u_i = \sum_{j=1}^{3} B_{ji} u'_j, \qquad (i = 1, 2, 3),$$

where B_{ij} is the cofactor of b_{ij} in the determinant $|b_{ij}|$.

THEOREM 1. *There exists a unique linear transformation of the lines of M into the lines of M' which carries four given lines, no three concurrent, in M respectively into four prescribed lines, no three concurrent, in M'.*

THEOREM 2. *The projective transformations and the linear transformations of the lines of M into the lines of M' are identical.*

EXERCISES

1. Prove that the transformation (1) is projective.

2. Prove Theorem 2.

3. The linear transformation of § 9, Ex. 1 transforms an arbitrary line u of the plane M into a line u' of the plane M'. Find the equations of this transformation of lines.

13. Collineations. The general projective transformation of the points of a plane M into those of a plane M',

$$\rho x'_i = \sum_{j=1}^{3} a_{ij} x_j, \quad (i = 1, 2, 3), \quad |a_{ij}| \neq 0, \tag{1}$$

establishes a transformation of the lines of M into those of M' in that it carries an arbitrary line u in M into a line u' in M'. To find the equations of this transformation of lines, we substitute the values of x_1, x_2, x_3 given by the inverse of (1):

$$\sigma x_i = \sum_{j=1}^{3} A_{ji} x'_j, \qquad (i = 1, 2, 3), \tag{1'}$$

into the equation,

$$u_1 x_1 + u_2 x_2 + u_3 x_3 = 0,$$

of the line u, and collect, in the resulting equation, the terms in x'_1, x'_2, x'_3. We thus obtain, as the equation of the line u',

$$x'_1 \sum_{j=1}^{3} A_{1j} u_j + x'_2 \sum_{j=1}^{3} A_{2j} u_j + x'_3 \sum_{j=1}^{3} A_{3j} u_j = 0,$$

and hence, as the coordinates of u',

$$\lambda u'_i = \sum_{j=1}^{3} A_{ij} u_j, \qquad (i = 1, 2, 3). \tag{2}$$

Equations (2) tell how the line u is transformed into the line u'. They are the equations of the required transformation.

Since (1) is the inverse of (1′), it follows by analogy that the inverse of (2) is

$$(2') \qquad \mu u_i = \sum_{j=1}^{3} a_{ji} u'_j, \qquad (i = 1, 2, 3).$$

Transformations (2) and (2′) are linear and therefore projective. Hence:

THEOREM 1. *A projective transformation of the points of M into the points of M' establishes a projective transformation of the lines of M into the lines of M'.*

Let the reader prove the dual theorem, showing that, if the equations of the given transformation of the lines of M into the lines of M' and its inverse are

$$(3) \qquad \rho u'_i = \sum_{j=1}^{3} b_{ij} u_j, \qquad \sigma u_i = \sum_{j=1}^{3} B_{ji} u'_j, \qquad (i = 1, 2, 3),$$

those of the induced transformation of the points of M into the points of M' and its inverse are

$$(4) \qquad \lambda x'_i = \sum_{j=1}^{3} B_{ij} x_j, \qquad \mu x_i = \sum_{j=1}^{3} b_{ji} x'_j, \qquad (i = 1, 2, 3).$$

It follows from these theorems that the two kinds of projective transformations of M into M' differ only in that in the one case the point, and in the other the line, is used as the fundamental element. In other words, *they differ only in point of view.* It is natural, then, to give them a common name. We call them *collineations.*

DEFINITION. *A collineation is a projective transformation of a plane into a plane which carries points into points and hence lines into lines, or lines into lines and hence points into points.*

The term "collineation" pertains especially neither to point geometry nor to line geometry, but equally to both. A collineation may be expressed either in point coordinates or in line coordinates. If (1) is the collineation in point coordinates, then (2) is the same collineation in line coordinates. Again, equations (3) and (4) represent the same collineation.

EXERCISES

1. Prove the dual of Theorem 1.

2. Find, using the methods but not the formulas of the text, the equations in line coordinates of the collineation

$$\rho x'_1 = x_2 + x_3, \qquad \rho x'_2 = x_1 + x_3, \qquad \rho x'_3 = x_1 + x_2.$$

14. Groups of Transformations and Their Associated Geometries. Suppose that M and M' are the same plane and that (x_1, x_2, x_3), (x'_1, x'_2, x'_3) are referred to the same coordinate system in this plane. The general collineation of the plane into itself, represented in point coordinates, is

$$(1) \qquad \rho x_i = \sum_{j=1}^{3} a_{ij}x_j, \qquad (i = 1, 2, 3), \qquad |a_{ij}| \neq 0.$$

The totality of these collineations form a group; see the text and Ex. 5 of § 9.

THEOREM 1. *All the collineations of a plane into itself form a group.*

The equations of the collineation (1) in nonhomogeneous coordinates are readily found to be

$$(2) \qquad x' = \frac{a_{11}x + a_{12}y + a_{13}}{a_{31}x + a_{32}y + a_{33}}, \qquad y' = \frac{a_{21}x + a_{22}y + a_{23}}{a_{31}x + a_{32}y + a_{33}}.$$

The general rigid motion of the plane,

$$(3) \quad x' = x \cos\theta - y \sin\theta + a, \qquad y' = x \sin\theta + y\cos\theta + b,$$

may be obtained from (2) by proper specialization of the coefficients a_{ij}. Hence:

THEOREM 2. *The group of rigid motions of the plane is a subgroup of the group of collineations.*

To bring out the significance of this theorem, let us recall our characterizations of metric and projective geometry. A projective property is one which is invariant with respect to the group of collineations, that is, projective transformations, and projective geometry consists of the study of projective properties. On the other hand, a property is metric if it is invariant with respect to the group of rigid motions, but not with respect to the larger group of collineations, and the study of metric properties, either in themselves or in conjunction with projective properties, constitutes metric geometry.

Thus, projective geometry may be described as the geometry associated with the group of collineations, and metric geometry as the geometry associated with the group of rigid motions.

Theorem 2 tells us that the group of rigid motions is a subgroup of the group of collineations. Accordingly, metric geometry must be capable of an interpretation whereby it appears as a "subgeometry" of projective geometry. The nature of this interpretation we shall discuss in the next chapter.

EXERCISES ON CHAPTER VII

1. *Transformations of Similarity.* The effect of a rigid motion is merely a change of position. The effect of a transformation of similarity may also involve a *uniform* enlargement or reduction of size. In other words, a rigid motion preserves both the shape and size of figures, whereas a transformation of similarity preserves shape but not necessarily size.

Show that the *radial transformation* from the origin

$$x' = \rho x, \qquad y' = \rho y, \qquad \rho > 0,$$

is a special transformation of similarity, and that the general transformation of similarity is the product of the general rigid motion and this radial transformation, and has therefore the equations

$$x' = \rho(x \cos\theta - y \sin\theta + a), \qquad y' = \rho(x \sin\theta + y \cos\theta + b).$$

Prove that the transformations of similarity form a group.

2. *Homothetic Transformations.* These are defined by the equations

$$x' = \rho x + a, \qquad y' = \rho y + b, \qquad \rho > 0.$$

Prove for them the following properties.

(*a*) A homothetic transformation is the product of a radial transformation from the origin and a translation, or either alone.

(*b*) A homothetic transformation is either a translation or a radial transformation from a fixed point.

(*c*) The homothetic transformations form a group.

3. The equation

$$x' = \frac{x+3}{x-1}$$

represents a projective transformation of a line into itself. Show that it leaves each of the points $x = 3$, $x = -1$ fixed in position, and carries any other point into its harmonic conjugate with respect to these fixed points.

4. A line is carried into itself by the linear transformation

$$x' = \frac{3x+2}{x+4}.$$

Prove that two points of the line remain fixed, and that every other point P and the point P' into which it is carried divide these fixed points in a constant cross ratio.

5. A transformation carries each point P of a line, other than two specific points P_1, P_2, into the point P' for which the cross ratio (P_1P_2, PP') is equal to a given constant k, $\neq 0, 1$. It carries each of the points P_1, P_2 into itself. Show that it is a projective transformation.

6. Prove that the collineation

$$x' = \frac{x}{2x-1}, \qquad y' = \frac{y}{2x-1}$$

leaves the origin O and each point on the line $x = 1$ fixed and carries any other point P into its harmonic conjugate with respect to O and the point in which OP meets $x = 1$. Show that the collineation is its own inverse.

7. Describe geometrically the collineation

$$x' = \frac{[illegible]}{3x - 2}, \qquad y' = \frac{[illegible]}{3x - 2}.$$

8. Find the equation of the line L' into which the line $L : y - 6 = 0$ is carried by the collineation

$$x' = \frac{3x + y + 4}{y - 1}, \qquad y' = \frac{-4x - y + 1}{y - 1}.$$

Show that, in this particular case, L is carried into L' as by a rigid motion, that is, so that distance is preserved.

9. Find the curve into which the conic

$$2x^2 + 3xy + 2y^2 + x + y - 1 = 0$$

is carried by the collineation

$$\rho x_1' = 2x_1 + x_2 + x_3, \qquad \rho x_2' = x_1 + 2x_2 + x_3, \qquad \rho x_3' = x_1 + x_2 + 2x_3.$$

10. *Singular Linear Transformations.*

(*a*) Show that the transformation

$$\rho x_1' = 2x_1 - x_2 + x_3, \qquad \rho x_2' = x_1 + x_2 + x_3, \qquad \rho x_3' = 4x_1 - 5x_2 + x_3$$

is *singular*, that is, that its determinant is zero. Prove that the transformation carries every point of the plane, except one, into a point of the line

$$3x_1' - 2x_2' - x_3' = 0.$$

What is the exceptional point? What is the geometric analog of the transformation?

(*b*) Show that the singular transformation

$$\rho x_1' = x_1 - x_2 + x_3, \qquad \rho x_2' = -2x_1 + 2x_2 - 2x_3, \qquad \rho x_3' = 3x_1 - 3x_2 + 3x_3$$

carries every point of the plane, other than those of a certain line, into the point $(1, -2, 3)$. Has the transformation a geometric analog?

(*c*) Show that the ranks of the matrices of the coefficients of the transformations of (*a*) and (*b*) are respectively 2 and 1. Formulate a general theorem concerning singular linear transformations and their geometric analogs.

CHAPTER VIII

METRIC GEOMETRY OF THE COMPLEX PLANE

1. Introduction. Complex Numbers. Thus far, we have treated primarily only linear problems, problems which, when formulated analytically, involve only linear equations. Let us now consider a quadratic problem, for example, that of finding the points of intersection of the circle and line

$$x^2 + y^2 = 1, \qquad x = a, \qquad a > 0.$$

If $a < 1$, the line intersects the circle in two points. But if $a > 1$, the line fails to meet the circle. We can, however, still solve the two equations. Thus, for $a = 2$, the solutions are $(2, \sqrt{-3})$, $(2, -\sqrt{-3})$, where the value of y in both cases is imaginary.

It is in accord with previous aims to demand that the line always meet the circle in two points. To meet this demand in the case $a = 2$, we create two ideal points to which we assign the number pairs $(2, \sqrt{-3})$, $(2, -\sqrt{-3})$ as coordinates.

Before proceeding to a systematic development of the idea thus suggested, we devote a few words to complex numbers.

Complex Numbers. If a' and a'' are arbitrary real numbers, the general complex number, a, is

$$a = a' + i\,a'',$$

where i is an ideal number which has the property that its square is $-1 : i^2 = -1$.

The *real* numbers are simply the complex numbers for which $a'' = 0$. On the other hand, if $a'' \neq 0$, the complex number $a = a' + i\,a''$ is called *imaginary*.

The terms "complex" and "imaginary" are not equivalent. "Imaginary" is the opposite of "real" and "complex" includes both.

The algebra of complex numbers is the same as that of real numbers. In particular, the theories of determinants, matrices, linear equations, and linear dependence, developed in Ch. I, remain unchanged when real numbers are everywhere replaced by complex numbers.

Exercise. Show that, if the product of two complex numbers, $a = a' + i\,a''$, $b = b' + i\,b''$, is zero, at least one of the numbers is zero.

2. Complex Points and Lines. *Points.* Let a Cartesian system of axes in the plane be given and let

$$x = x' + i\,x'', \qquad y = y' + i\,y''$$

be a pair of complex numbers. If x and y are both real, they are the coordinates of an ordinary point, or a *real point,* as it shall henceforth be called. If x, or y, or both, are imaginary, there shall be created a new point which shall be known as an *imaginary point,* and to which (x, y) shall be given as coordinates.

The totality of points (x, y), real and imaginary, constitutes the *finite complex plane.* An arbitrary point of this plane shall be called a *complex point.* A complex point is, then, like a complex number, either real or imaginary.

Instead of the nonhomogeneous coordinates (x, y), we can use homogeneous coordinates:

$$x_1 = x_1' + i\,x_1'', \qquad x_2 = x_2' + i\,x_2'', \qquad x_3 = x_3' + i\,x_3''.$$

If x_1, x_2, x_3 are proportional to a triple of real numbers, other than 0, 0, 0, they are the coordinates of a real point; thus, $(2, 0, 4)$, $(2\,i, 0, 4\,i)$, $(2 - 2\,i, 0, 4 - 4\,i)$ are all sets of coordinates for the same real point. If the triple x_1, x_2, x_3 is not proportional to a triple of real numbers, other than 0, 0, 0, it and every proportional triple $(\rho x_1, \rho x_2, \rho x_3)$, where ρ is an arbitrary complex number, not zero, are sets of homogeneous coordinates of a newly created imaginary point. The totality of complex points, that is, of real and imaginary points, (x_1, x_2, x_3), forms the *extended complex plane.*

Lines. The introduction of imaginary lines is analogous to that of imaginary points. We have merely to employ line coordinates where before we used point coordinates.

Points and Lines. The condition that the real point x lie on the real line u is

$$(1) \qquad (u\,|\,x) \equiv u_1x_1 + u_2x_2 + u_3x_3 = 0.$$

We agree to take this also as the condition that the complex point x lie on the complex line u.

The agreement implies that the theory of the relationship between point and line, as developed for the real plane in Ch. V, § 4, holds equally well for the complex plane. From it follows that $(a\,|\,u) = 0$ is the equation of the complex point a and that $(a\,|\,x) = 0$ is the equation of the complex line a, and from these facts the validity in the complex plane of the entire theory in question can be established.

EXERCISES

1. Show that the line which joins the points $(1, i, 3 + 2i)$, $(1, -i, 3 - 2i)$ is a real line.

2. Find the point of intersection of the lines

$$2x_1 - ix_2 + (1 + i)x_3 = 0, \qquad 2ix_1 - x_2 + (1 + i)x_3 = 0.$$

3. Conjugate-Complex Elements. The two numbers

$$a = a' + ia'', \qquad \bar{a} = a' - ia''$$

are known as *conjugate-complex* numbers. In particular, if $a'' \neq 0$, they are called *conjugate-imaginary* numbers.

THEOREM 1. *A necessary and sufficient condition that a complex number be real is that it be equal to the conjugate-complex number.*

For, if $a'' = 0$, then $a = \bar{a}$. Conversely, if $a = \bar{a}$, then $ia'' = 0$ and $a'' = 0$.

DEFINITION. *Two elements, that is, two points or two lines, are conjugate-complex if, when (a_1, a_2, a_3) are homogeneous coordinates of one, $(\bar{a}_1, \bar{a}_2, \bar{a}_3)$ are homogeneous coordinates of the other.*

The definition implies that two finite elements are conjugate-complex when and only when their corresponding *nonhomogeneous* coordinates are conjugate-complex. On the other hand, corresponding *homogeneous* coordinates of two conjugate-complex elements are not necessarily conjugate-complex, inasmuch as homogeneous coordinates admit a factor of proportionality.*

THEOREM 2. *An element is real if and only if it coincides with the conjugate-complex element.*

Homogeneous coordinates of a real element can be taken as real numbers; the conjugate-complex element has, then, the same coordinates and hence coincides with the given element.

Suppose, conversely, that two conjugate-complex elements are identical. Homogeneous coordinates (a_1, a_2, a_3) can be chosen for one of them so that one of the coordinates, say a_3, is unity: $a_3 = 1$. Coordinates of the other element are, then, $(\bar{a}_1, \bar{a}_2, \bar{a}_3)$. where $\bar{a}_3 = 1$. Since the two are identical,

$$a_1 = \rho\bar{a}_1, \qquad a_2 = \rho\bar{a}_2, \qquad a_3 = \rho\bar{a}_3.$$

* The elements $(2, i, 1 - i)$ and $(2 + 2i, 1 - i, 2i)$ are conjugate-imaginary, despite the fact that the corresponding coordinates are not conjugate-imaginary; for, alternative coordinates are $(2, i, 1 - i)$ and $(2, -i, 1 + i)$.

But $a_3 = \bar{a}_3 = 1$; hence $\rho = 1$ and $a_1 = \bar{a}_1$, $a_2 = \bar{a}_2$. Therefore, a_1 and a_2 are real and the element is real. The proof of the theorem is thus complete.

The expression $(u|x)$ can be considered as a single complex number. The conjugate-complex number is $(\bar{u}|\bar{x})$; see Ex. 2. But if a complex number is zero, the conjugate-complex number is zero. Therefore, if

$$u_1x_1 + u_2x_2 + u_3x_3 = 0,$$

then

$$\bar{u}_1\bar{x}_1 + \bar{u}_2\bar{x}_2 + \bar{u}_3\bar{x}_3 = 0.$$

THEOREM 3. *If a point x lies on a line u, the conjugate-complex point $\bar{x}$ lies on the conjugate-complex line $\bar{u}$.*

In particular, if an imaginary point lies on a real line, the conjugate-imaginary point lies on this line. Hence, the points of a real line are either real or conjugate-imaginary in pairs. Similarly, the lines through a real point are real or conjugate-imaginary in pairs.

We are now in a position to prove the following important proposition.

THEOREM 4. *The point of intersection of two conjugate-imaginary lines is a real point. The line determined by two conjugate-imaginary points is a real line.*

Let L and $\bar{L}$ be two conjugate-imaginary lines, and let P be their point of intersection. By Th. 3, since P lies on L and $\bar{L}$, the conjugate-complex point, $\bar{P}$, lies on $\bar{L}$ and L. Hence $\bar{P}$ is identical with P, and therefore, by Th. 2, P is real.

We give an analytical proof of the second half of the theorem. Let the two conjugate-imaginary points be a and $\bar{a}$. Coordinates of the line L joining them are

$$a_2\bar{a}_3 - a_3\bar{a}_2, \qquad a_3\bar{a}_1 - a_1\bar{a}_3, \qquad a_1\bar{a}_2 - a_2\bar{a}_1.$$

The conjugates of these, which, by Ex. 1, reduce to

$$\bar{a}_2a_3 - \bar{a}_3a_2, \qquad \bar{a}_3a_1 - \bar{a}_1a_3, \qquad \bar{a}_1a_2 - \bar{a}_2a_1,$$

are coordinates of the conjugate-complex line $\bar{L}$. But these coordinates are the negatives of the preceding ones. Hence, L coincides with $\bar{L}$ and is a real line.

COROLLARY. *There is one real line through an imaginary point and one real point on an imaginary line.*

A real point on an imaginary line is the point in which it intersects the conjugate-imaginary line. If there were a second real point, the line would be real.

EXERCISES

1. Show that, if a and b are two complex numbers, the conjugate of their sum equals the sum of their conjugates and the conjugate of their product equals the product of their conjugates:

$$\overline{a+b} = \bar{a} + \bar{b}, \qquad \overline{ab} = \bar{a}\bar{b}.$$

Extend the theorem to the case of any finite number of complex numbers.

2. By means of Ex. 1, prove that the conjugate of $(u|x)$ is $(\bar{u}|\bar{x})$.

3. Show that, if a and b are conjugate-complex numbers, their sum and product are real. How must the statement be modified in order that the converse be true?

4. Show that the two lines of § 2, Ex. 2 are conjugate-imaginary.

5. Find the real point on the line $(2, i, 3 - 4i)$.

6. What is the real line through the point $(1, i, 0)$?

7. Prove that the three points $(1 + i, -1 + i)$, $(1, 1 + i)$, $(i, -1 - i)$ are collinear. Find the real point on their line.

8. Give a geometrical proof of the second part of Theorem 4.

9. Prove that the real point on an imaginary line is the point at infinity on the line if and only if the line has a real slope or is parallel to the y-axis.

10. Show that the line determined by two given points and the line determined by the two conjugate-complex points are conjugate-complex.

4. Metric Geometry of the Complex Plane. Let a circle, with center at O and radius r, be given. Let A be any point in the plane of the circle, other than the center. On the half-line OA, take the point B so that

$$OA \cdot OB = r^2.$$

At C, the mid-point of the segment AB, erect a perpendicular meeting the circle in D and draw AD and OD.

From the right triangles ACD and OCD,

$$AD^2 = CD^2 + AC^2, \qquad CD^2 = r^2 - OC^2.$$

Hence

$$AD^2 = r^2 - (OC^2 - AC^2),$$
$$AD^2 = r^2 - (OC - AC)(OC + AC),$$
$$AD^2 = r^2 - OA \cdot OB = r^2 - r^2 = 0,$$

and

$$AD = 0.$$

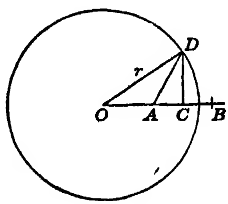


FIG. 1

The distance from A to D is zero. Therefore A coincides with D: every point, other than the center of the circle, lies on the circumference of the circle!

We invite the reader to lay the book aside and find for himself the fallacy.

Critique. Because of the use of the figure, the reader doubtless assumed that the proof is restricted to the case of a real circle and a real point A. He may, then, make the shrewd and correct guess that the point C lies without the circle, so that the point D is imaginary. But is he then any better off? Would he want to conclude that the real point A is identical with the imaginary point D?

As a matter of fact, the deduction of the fact that $AD = 0$ holds, in general, regardless of whether the circle and the point A are real. One or both of them may be imaginary. The figure is, in that case, simply symbolic; it merely suggests the steps to be taken in the deduction and is not in any way a part of the deduction.

In working in the complex plane we ought to be wary of assuming that a fact which is true for real points is also true for imaginary points. We might well question whether the Pythagorean theorem, on which the deduction that $AD = 0$ is based, is always true in the complex plane. As a matter of fact, it is true, as will be proved later.

We are thus forced to conclude that the proof that $AD = 0$ is always valid. There remains, then, but one question. Does A coincide with D? Are two complex points, which are zero distance apart, necessarily coincident?

EXERCISES

1. Find all the solutions of the equation

$$x^2 + y^2 = 0,$$

if x and y are restricted to be real? If x and y are complex? Interpret the results geometrically.

2. Explain the fallacy in the text and verify your explanation analytically, assuming that the circle and the point A are real and taking O as the origin and OA as the axis of x.

5. Isotropic Lines. We proceed to a systematic discussion of distance and angle in the complex plane. The formulas for them we assume to be the same as in the real plane.

Distance between Two Points. If the points are (x_1, y_1), (x_2, y_2), the square of the distance is

$$D^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2.$$

There are, in general, two values for the distance D, and there is usually no reason why one of the values should be preferred to the

other.* Accordingly, we shall restrict ourselves, whenever possible, to the square of the distance.

The one case when D is single-valued occurs when D^2 is zero. This case, though trivial in the real plane, is now, in view of the paradox of the previous paragraph, of peculiar interest to us.

Let us investigate the locus of a point (x, y) which moves always at zero distance from the fixed point (x_0, y_0). The equation of the locus is

$$(x - x_0)^2 + (y - y_0)^2 = 0.$$

But

$$(x - x_0)^2 + (y - y_0)^2 = [(x - x_0) - i(y - y_0)][(x - x_0) + i(y - y_0)].$$

Hence the locus consists of the two straight lines

(1) $\quad (x - x_0) - i(y - y_0) = 0, \qquad (x - x_0) + i(y - y_0) = 0$

which pass through (x_0, y_0) and have respectively the slopes $-i$ and i.†

A line whose slope is $-i$ or i we shall call an *isotropic line*, or simply an *isotropic*. The answer to our locus problem then becomes

Theorem 1. *The locus of a point which moves so that it is always at zero distance from a given point consists of the two isotropic lines through the given point.*

Any two finite points P_1, P_2 on an isotropic L are at zero distance; for, L is part of the locus of points at zero distance from P_1 and therefore, since P_2 lies on L, $P_1P_2 = 0$. Conversely, if P_1, P_2 are at zero distance, the line joining them is an isotropic; for, since $P_1P_2 = 0$, P_2 lies on the locus of points at zero distance from P_1, and hence lies on one of the isotropics through P_1.

Theorem 2. *A necessary and sufficient condition that a line be an isotropic is that the distance between any two finite points on it be zero.*

The isotropics of slope $-i$ form the pencil of parallel lines through the point at infinity $I : (1, -i, 0)$; and the isotropics of slope i, the

* If the points, for example, are (i, i), $(1, 1)$, then $D^2 = 2(1 - i)^2$ and $D = \pm\sqrt{2}\,(1 - i)$.

† We agree to employ the usual formula,

$$\lambda = \frac{y_2 - y_1}{x_2 - x_1},$$

for the slope of a line. It follows that the slope of a line can be read from its equation in the usual way.

pencil of parallel lines through the point at infinity $J : (1, i, 0)$. The equations of the two pencils are

$$x - iy + k = 0, \qquad x + iy + l = 0,$$

where k and l are arbitrary constants.

The Circular Points at Infinity. The null circle,

$$(x - x_0)^2 + (y - y_0)^2 = 0,$$

inasmuch as it consists of the isotropic lines (1), goes through the points I and J. In fact, every circle goes through I and J, for, the equation of an arbitrary circle, in homogeneous coordinates, is

$$x_1^2 + x_2^2 + a_1x_1x_3 + a_2x_2x_3 + a_3x_3^2 = 0,$$

and it is readily verified that $(1, -i, 0)$ and $(1, i, 0)$ satisfy this equation.

Conversely, if the conic,

$$Ax_1^2 + Bx_1x_2 + Cx_2^2 + Dx_1x_3 + Ex_2x_3 + Fx_3^2 = 0,$$

contains I and J, then *

$$A - Bi - C = 0, \qquad A + Bi - C = 0,$$

whence $A = C$ and $B = 0$: the conic is a circle.

THEOREM 3. *A necessary and sufficient condition that a conic be a circle is that it pass through the vertices, I and J, of the pencils of isotropic lines.*

It is from this property that I and J take their conventional name: *the circular points at infinity.*

Distance from a Point to a Line. The square of the distance is given by the usual formula:

$$d^2 = \frac{(a_1x_0 + a_2y_0 + a_3)^2}{a_1^2 + a_2^2}.$$

The distance is, in general, defined and double-valued. However, when the line is an isotropic, then $a_1^2 + a_2^2 = 0$ and the formula has no meaning. *The distance from an isotropic line to a point, even if the point is on the line, is undefined.*

Angle and Perpendicularity. Let L_1 and L_2 be two isotropics of the same kind, that is, two isotropics whose slopes, λ_1 and λ_2, have the same value, i or $-i$. Then $\lambda_1\lambda_2 = -1$, or $1 + \lambda_1\lambda_2 = 0$. Since this is the usual condition for perpendicularity, we are tempted to say

* It is assumed that A, B, C are not all zero.

that L_1 and L_2, which are already known to be parallel, are also perpendicular!

According to the agreement made at the beginning of the paragraph, the angle from one line to another is given by the usual formula

$$\tan\theta = \frac{\lambda_2 - \lambda_1}{1 + \lambda_1\lambda_2}, \tag{2}$$

or by the corresponding formula for $\cot\theta$, so long as either has meaning. In the present case, $\lambda_1 = \lambda_2$, $1 + \lambda_1\lambda_2 = 0$, and neither formula has meaning: *the angle θ is undefined.*

Accordingly, we cannot talk of the angle between two isotropic lines of the same kind. Nor can we talk of their being perpendicular, for perpendicularity means intersection under a right angle. The two isotropics remain simply what they were originally—parallel.*

We consider next the angle from an isotropic to any other finite line, not an isotropic of the same kind. If the isotropic is of slope $-i$ and the second line of slope λ, we have

$$\tan\theta = \frac{\lambda + i}{1 - \lambda i} = i\,\frac{\lambda + i}{\lambda + i} = i.$$

Now †

$$\tan\theta = \frac{1}{i}\,\frac{e^{\theta i} - e^{-\theta i}}{e^{\theta i} + e^{-\theta i}}. \tag{3}$$

When $\tan\theta = i$, this relation becomes

$$e^{\theta i} = 0, \qquad \text{or} \qquad \theta i = \log 0!$$

If we had taken an isotropic of slope i, we should have found that $\tan\theta = -i$ and hence that

$$e^{-\theta i} = 0, \qquad \text{or} \qquad -\theta i = \log 0!$$

In both cases, the angle θ is undefined. Hence we conclude:

An isotropic line fails to make an angle with any line.

* The concept of parallelism is independent of that of angle. Two finite lines are parallel, if they intersect in a point at infinity.

† Assuming the relations,

$$e^{\theta i} = \cos\theta + i\sin\theta, \qquad e^{-\theta i} = \cos\theta - i\sin\theta, \tag{a}$$

we obtain, on addition and subtraction,

$$\sin\theta = \frac{e^{\theta i} - e^{-\theta i}}{2i}, \qquad \cos\theta = \frac{e^{\theta i} + e^{-\theta i}}{2}. \tag{b}$$

Thus, $\tan\theta$ has the value given in (3). See Osgood, *Advanced Calculus*, pp. 337, 338 for a discussion of the relations (a) when θ is real and *ibid.*, pp. 503, 504 for the justification of equations (b) and (3) when θ is complex.

The converse is true: *If the angle between two finite lines fails to exist, at least one of the lines is an isotropic.* The angle between two lines, of slopes λ_1 and λ_2, fails to exist when $\tan\theta$, as given by (2), is of the form 0/0, or when $\tan\theta$, or $\cot\theta$, is defined, but θ, as given by (3) or the equivalent formula

$$(4) \qquad \theta = \frac{1}{2i}\log\frac{i - \tan\theta}{i + \tan\theta},$$

is not defined. If $\tan\theta = 0/0$, then $\lambda_1 = \lambda_2$ and $1 + \lambda_1\lambda_2 = 0$, and the two lines are isotropics of the same kind. On the other hand, it is a fact, the proof of which need not concern us, that θ, as given by (4), is undefined only if $\tan\theta = i$ or $\tan\theta = -i$. Then (2) reduces to

$$(\lambda_1 + i)(\lambda_2 - i) = 0 \qquad \text{or} \qquad (\lambda_1 - i)(\lambda_2 + i) = 0,$$

and at least one of the lines is an isotropic.

Remarks. Inasmuch as an isotropic line fails to make an angle with any line, it has no slope-angle. This does not mean that it has no slope; for the slope λ of a line is given by the familiar formula,

$$\lambda = \frac{y_2 - y_1}{x_2 - x_1},$$

and, thus defined, it is no longer dependent on the concept of angle; it is the ratio of the projections on the axes of a directed line-segment on the line.

The tangent of the angle θ from a line L_1 to a line L_2 can also be defined independently of the angle θ. In fact, a slight extension of the usual definition of $\tan\theta$, as the ratio of two directed line-segments (Fig. 2), yields

$$\tan\theta \qquad \frac{MP_2}{P_1M}$$

from which the usual formula for $\tan\theta$ in terms of the slopes of L_1 and L_2 can be readily derived.*

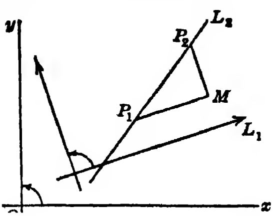


FIG. 2

Since the concept of the tangent of the angle from L_1 to L_2 is a generalization of the idea of slope, we can justifiably call it the *relative*

* It is understood, of course, that we derive the formula for real lines and then agree to apply it also to imaginary lines.

slope of L_2 with respect to L_1. Thereby, the name as well as the definition is divorced from the idea of angle.

The angle from an isotropic to a second line is undefined. But the relative slope of an isotropic with respect to any finite line, not an isotropic of the same kind, is always defined. We found for it the constant value, i or $-i$, according as the isotropic is of slope $-i$ or i. Hence, *an isotropic has the same relative slope with respect to every finite line, not an isotropic of the same kind.* It is from this property that the isotropic lines take their name.*

Isotropic lines are frequently called *minimal* lines, since they are the lines of zero distance and hence, so it is argued, of minimum distance. The terminology is, however, not well motivated. For, though zero is the minimum distance in the real plane, it cannot be so regarded in the complex plane. There is no "greater than" or "less than" for complex numbers and, therefore, no smallest complex number.

EXERCISES

1. Prove directly, by means of the distance formula, that the distance between two finite points on an isotropic of slope $-i$ is zero.

2. Find the points on the line $x + y + 2 = 0$ which are at zero distance from the origin.

3. Prove that the finite line $a_1x + a_2y + a_3 = 0$ is an isotropic if and only if $a_1^2 + a_2^2 = 0$.

4. If an ordinary quadrilateral has as its sides four isotropic lines, two of each kind, the diagonals are mutually perpendicular and bisect one another. Prove the theorem in the special case in which two opposite vertices of the quadrilateral are the conjugate-imaginary points $(2, 3i)$ and $(2, -3i)$.

5. Show that the square of the distance between two conjugate-imaginary points is real and negative.

6. Show that two conjugate-imaginary lines can never be perpendicular.

7. Show that the Pythagorean theorem is valid in the complex plane.

8. What is the locus of a point which moves so that it always subtends a right angle at two given imaginary points whose join is an isotropic line? Substantiate your answer analytically.

6. Laguerre's Definition of Angle. The group of rigid motions possesses, as an invariant, the square of the distance between two points, real or imaginary. A rigid motion must, therefore, carry two points at zero distance into two points at zero distance and hence an isotropic into an isotropic.

* Derived from the Greek: ἴσος, *equal,* and τρέπω, *to turn.*

To prove this analytically, we have merely to show that a rigid motion carries the circular points at infinity into themselves, since then it will carry a finite line through a circular point into a finite line through a circular point, that is, an isotropic into an isotropic.

The equations, in homogeneous coordinates, of the general rigid motion are

$$\begin{aligned} \rho x_1' &= x_1 \cos\theta - x_2 \sin\theta + a\,x_3, \\ \rho x_2' &= x_1 \sin\theta + x_2 \cos\theta + b\,x_3, \\ \rho x_3' &= \qquad\qquad\qquad\qquad\quad x_3. \end{aligned}$$

Substituting for (x_1, x_2, x_3) the coordinates $(1, -i, 0)$ of I, we get $x_3' = 0$ and

$$\begin{aligned} \rho x_1' &= \cos\theta + i \sin\theta = \quad 1\,(\cos\theta + i\sin\theta), \\ \rho x_2' &= \sin\theta - i\cos\theta = -i\,(\cos\theta + i\sin\theta). \end{aligned}$$

Taking ρ equal to $\cos\theta + i\sin\theta$, we have $x_1' = 1$, $x_2' = -i$. Hence the rigid motion carries I into itself. Similarly, it carries J into itself.

THEOREM 1. *Every rigid motion carries each of the circular points at infinity into itself, and each isotropic line into an isotropic line of the same kind.*

Two finite nonisotropic lines, L_1 and L_2, intersecting in a finite point P, and the isotropics L_I and L_J through P are carried respectively into two finite nonisotropic lines, L_1' and L_2', intersecting in a finite point P', and the isotropics L_I' and L_J' through P'. A rigid motion, being a special collineation, preserves cross ratio. Hence

$$(L_1'L_2',\, L_I'L_J') = (L_1L_2,\, L_IL_J).$$

THEOREM 2. *The cross ratio,*

$$r = (L_1L_2,\, L_IL_J), \tag{1}$$

in which the nonisotropic lines L_1, L_2 are separated by the isotropic lines L_I, L_J through their point of intersection is an invariant with respect to the group of rigid motions.

Inasmuch as, when L_1 and L_2 are chosen, the isotropics L_I and L_J are determined, the cross ratio r is an invariant of the lines L_1, L_2 alone. But the angle θ from L_1 to L_2 is a metric invariant of L_1, L_2, and we should hardly expect to find a second invariant which is independent of θ. It is natural, then, to look for a relationship connecting r and θ.

Suppose that we compute r. Let the slopes of L_1 and L_2 be λ_1

and λ_2.* Those of L_I and L_J are $-i$ and i. Hence

$$r = \frac{(\lambda_1 + i)(\lambda_2 - i)}{(\lambda_1 - i)(\lambda_2 + i)} = \frac{(1 + \lambda_1\lambda_2) + i(\lambda_2 - \lambda_1)}{(1 + \lambda_1\lambda_2) - i(\lambda_2 - \lambda_1)}.$$

But

$$\tan\theta = \frac{\lambda_2 - \lambda_1}{1 + \lambda_1\lambda_2}.$$

Hence

$$r = \frac{1 + i\tan\theta}{1 - i\tan\theta}$$

or

(2)
$$\tan\theta = \frac{1}{i}\frac{r - 1}{r + 1}.$$

Comparing this relation with

$$\tan\theta = \frac{1}{i}\frac{e^{2\theta i} - 1}{e^{2\theta i} + 1},$$

we have

(3)
$$r = e^{2\theta i}, \qquad \text{or} \qquad \theta = \frac{1}{2i}\log r.$$

THEOREM 3. *If L_1 and L_2 are finite nonisotropic lines intersecting in a finite point, the angle θ from L_1 to L_2 is given by*

$$\tan\theta = \frac{1}{i}\frac{r - 1}{r + 1} \qquad \text{or} \qquad \theta = \frac{1}{2i}\log r,$$

where r is the cross ratio $(L_1L_2,\ L_IL_J)$.

If L_1 and L_2 are perpendicular, it follows from (2) that $r = -1$, and conversely.

THEOREM 4. *Two finite nonisotropic lines are mutually perpendicular if and only if they are separated harmonically by the isotropic lines through their point of intersection.*

These theorems † constitute *projective definitions of angle and perpendicularity;* they characterize angle and perpendicularity in terms of cross ratio and harmonic division. They thus serve as a connecting link between metric and projective geometry, the importance of which, as will be borne in on us later, can hardly be overestimated.

They also throw light on the rôles of the isotropic lines. Formerly, we could only look on the isotropics as destructive; they upset not a

* The case in which one of the lines is parallel to the axis of y is left to the reader.

† The theorems were first established by the French geometer, Laguerre, in 1853.

few of our preconceptions. Now, the isotropics appear in a constructive rôle, as the basis for new interpretations of angle and perpendicularity.

Moreover, the peculiar behavior of an isotropic with respect to angle and relative slope now ceases to be a mystery. When one of the lines L_1, L_2 is an isotropic, the four lines in the cross ratio r are no longer distinct and we expect something peculiar (Ch. VI, § 7). Suppose, for example, that L_1 is the isotropic through P of slope $-i$, coincident therefore with L_I. Then $r = (L_I L_2,\ L_I L_J) = 0$. Thus, the cross ratio is the same, no matter what line, other than L_I itself, is chosen as L_2. Consequently, it is no wonder that the relative slope of L_I with respect to L_2 is constant.

EXERCISES

1. Prove directly that a rigid motion carries an isotropic of slope i into an isotropic of slope i.

2. Establish the theorem of § 5, Ex. 4 in the general case. Proceed synthetically, employing the harmonic properties of a complete quadrilateral and Laguerre's definition of perpendicularity.

7. The Relationship of Metric to Projective Geometry. We know that the rigid motions are particular collineations and we have just learned that they leave each of the circular points at infinity fixed in position. Can we characterize them as the collineations with this property?

If the collineation

$$(1)\qquad \begin{aligned} \rho x_1' &= a_{11}x_1 + a_{12}x_2 + a_{13}x_3, \\ \rho x_2' &= a_{21}x_1 + a_{22}x_2 + a_{23}x_3, \\ \rho x_3' &= a_{31}x_1 + a_{32}x_2 + a_{33}x_3, \end{aligned} \qquad |a_{ij}| \neq 0$$

leaves I and J fixed, it carries the line of I and J into itself: $x_3 = 0$ into $x_3' = 0$. Hence, $a_{31} = a_{32} = 0$ and $a_{33} \neq 0$. Dividing each of the equations (1) by a_{33}, we get

$$(2)\qquad \begin{aligned} \sigma x_1' &= a_1x_1 + a_2x_2 + a_3x_3, \\ \sigma x_2' &= b_1x_1 + b_2x_2 + b_3x_3, \\ \sigma x_3' &= \phantom{a_1x_1 + a_2x_2 + {}} x_3, \end{aligned} \qquad \Delta = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \neq 0,$$

where we have written σ in place of ρ/a_{33} and, for example, a_1 instead of a_{11}/a_{33}.

Since (2) is to carry each of the circular points at infinity into itself,

$$\sigma_1 = a_1 - i\,a_2, \qquad \sigma_2 = a_1 + i\,a_2,$$
$$-i\,\sigma_1 = b_1 - i\,b_2, \qquad i\,\sigma_2 = b_1 + i\,b_2.$$

Hence

$$i\,a_1 + a_2 + b_1 - i\,b_2 = 0,$$
$$-i\,a_1 + a_2 + b_1 + i\,b_2 = 0,$$

and

$$a_2 = -b_1, \qquad b_2 = a_1.$$

Substituting these results in (2) and introducing nonhomogeneous coordinates, we have

$$\text{(3)} \qquad \begin{aligned} x' &= a_1x - b_1y + a_3, \\ y' &= b_1x + a_1y + b_3, \end{aligned} \qquad \Delta = a_1^2 + b_1^2 \neq 0.$$

Since $a_1^2 + b_1^2 \neq 0$, we may set

$$\frac{a_1}{r} = \cos\theta, \quad \frac{a_3}{r} = a, \qquad \frac{b_1}{r} = \sin\theta, \quad \frac{b_3}{r} = b, \qquad \text{where} \quad r = \sqrt{a_1^2 + b_1^2}.$$

Equations (3) then take the form,

$$\text{(4)} \qquad \begin{aligned} x' &= r\,(x\cos\theta - y\sin\theta + a), \\ y' &= r\,(x\sin\theta + y\cos\theta + b), \end{aligned} \qquad \Delta = r^2, \; r > 0,$$

of a transformation of similarity; see Ex. 1, End of Ch. VII.

The answer to our question is now clear.

THEOREM 1. *The transformations of similarity are the collineations which leave each of the circular points at infinity fixed.**

If the transformation (4) carries the points (x_1, y_1), (x_2, y_2) into the points (x_1', y_1'), (x_2', y_2'), then

$$(x_2' - x_1')^2 + (y_2' - y_1')^2 = r^2[(x_2 - x_1)^2 + (y_2 - y_1)^2].$$

In other words, the square of the distance between two points is reproduced, not *absolutely* by the transformation, but *relatively*, with the factor of proportionality r^2. We say, then, that

$$(x_2 - x_1)^2 + (y_2 - y_1)^2$$

is a *relative invariant* of the points (x_1, y_1), (x_2, y_2) with respect to the group of transformations of similarity.

* We have proved only that a collineation which leaves I and J fixed is a transformation of similarity. Let the reader establish the converse.

In order that the square of the distance be an *absolute invariant*, r^2 must be equal to unity, that is, our group of transformations must reduce to the group of rigid motions.

THEOREM 2. *The rigid motions are the collineations which leave each of the points I and J fixed and have the square of the distance between two finite points not merely as a relative, but as an absolute, invariant.*

Rigid motions and transformations of similarity differ in only one particular. The former preserve both the shape and the size of a figure, whereas the latter preserve only the shape. The geometry of similarity, the geometry associated with the group of transformations of similarity, takes cognizance only of the shapes of figures, whereas metric geometry, the geometry of the group of rigid motions, also considers their dimensions.

The geometry of Euclid is, in part, a geometry of similarity and, in part, a metric geometry. In the theory of similar figures, it takes account only of shape; in the theory of congruent figures and in the large body of work on mensuration, it is concerned also with size. Taken as a *whole*, however, Euclidean geometry is metric geometry, inasmuch as it is only with respect to the group of rigid motions that all the properties studied in Euclidean geometry are preserved.

On the other hand, the distinction between metric geometry and the geometry of similarity is so slight as to be relatively unimportant, particularly in comparison with the relationships which the two geometries bear to projective geometry, the geometry associated with the group of all collineations. These relationships are evident from Theorems 1 and 2. The geometry of similarity is precisely the geometry of the subgroup of the group of collineations which consists of those collineations leaving the circular points at infinity fixed. Metric geometry is the geometry of this subgroup subject to the condition that distance be an absolute, instead of a relative, invariant. *Disregarding this condition as of minor significance, we can describe metric geometry as a "subgeometry" of projective geometry, obtained from it by demanding that two conjugate-imaginary points remain fixed.*

In light of this interpretation of metric geometry, the isotropics—the lines through the fixed points—and Laguerre's definitions of angle and perpendicularity appear in true perspective, as fundamental, rather than unnatural or artificial, elements in the theory.

EXERCISE. Determine the collineations which leave each of the two real points (1, 1, 0), (1, − 1, 0) fixed. Show that

$$(x_2 - x_1)^2 - (y_2 - y_1)^2$$

is a relative invariant of (x_1, y_1), (x_2, y_2) with respect to these collineations.

8. Applications to Conics. An equation of the second degree,

$$(1) \qquad Ax_1^2 + Bx_1x_2 + Cx_2^2 + Dx_1x_3 + Ex_2x_3 + Fx_3^2 = 0,$$

with real coefficients, of which A, B, C are not all zero, represents a conic. According as the *discriminant*

$$\Delta = F(4\,AC - B^2) + BDE - AE^2 - CD^2$$

does not, or does, vanish, the conic is nondegenerate or degenerate; and according as $B^2 - 4\,AC > 0$, $= 0$, or < 0, the conic is a hyperbola, parabola, or ellipse.

In the elliptic case, (1) can be reduced by a change of axes to one of the three forms

$$(2) \qquad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \qquad \frac{x^2}{a^2} + \frac{y^2}{b^2} = -1, \qquad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 0.$$

The second of these equations has no locus in the real plane. But in the complex plane, it has a locus comparable with that of the first equation. Thus, *there are two kinds of nondegenerate ellipses: those with real traces—continua of real points; and those without real traces.*

The degenerate ellipse represented by the third equation of (2) consists, for us, of the two conjugate-imaginary intersecting lines

$$(3) \qquad \frac{x}{a} + i\frac{y}{b} = 0, \qquad \frac{x}{a} - i\frac{y}{b} = 0.$$

In the parabolic case, equation (1) can be reduced to

$$y^2 = 2\,mx \qquad \text{or} \qquad y^2 = k,$$

according as $\Delta \neq 0$ or $\Delta = 0$. There is but one type of nondegenerate parabola. A degenerate parabola consists of two parallel lines, which may be real and distinct, real and coincident, or conjugate-imaginary.

There is but one type of nondegenerate, and one type of degenerate, hyperbola:

$$(4) \qquad \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1, \qquad \frac{x^2}{a^2} - \frac{y^2}{b^2} = 0.$$

A degenerate hyperbola consists of two real intersecting lines.

From this review of the facts we conclude the following theorem.

THEOREM 1. *A conic is degenerate if and only if it consists of two straight lines, distinct or coincident.*

Relationship of a Conic to the Line at Infinity. The points in which the conic (1) is met by the line at infinity, $x_3 = 0$, are given by the solutions of the quadratic equation

$$Ax_1^2 + Bx_1x_2 + Cx_2^2 = 0.$$

These points are real and distinct, real and coincident, or conjugate-imaginary, according as $B^2 - 4AC > 0, = 0$, or < 0.

THEOREM 2. *A conic is a hyperbola, a parabola, or an ellipse according as the two points in which it intersects the line at infinity are real and distinct, real and coincident, or conjugate-imaginary.*

The characterization of the parabola indicates that the nondegenerate * parabola,

$$y^2 = 2mx \qquad \text{or} \qquad x_2^2 = 2mx_1x_3,$$

is tangent to the line at infinity. To check this fact, we write the equation of the tangent at the point (x_0, y_0) or (r_1, r_2, r_3), namely

$$y_0y = m(x + x_0) \qquad \text{or} \qquad r_2x_2 = m(r_3x_1 + r_1x_3).$$

The point at infinity on the parabola is the point (1, 0, 0), in the direction of the axis. The tangent at this point is

$$0\,x_2 = m(0\,x_1 + 1\,x_3),\dagger \qquad \text{or} \qquad x_3 = 0.$$

THEOREM 3. *A parabola is tangent to the line at infinity at the point at infinity in the direction of its axis.*

If the conjugate-imaginary points in which an ellipse meets the line at infinity are, in particular, the circular points at infinity, the ellipse is a circle.

Asymptotes. It is reasonable to expect that the tangents to a hyperbola at the two points in which it intersects the line at infinity are the asymptotes of the hyperbola. Let the reader show that this is, in fact, the case.

On the basis of this result, we formulate a new definition of an asymptote.

* From now on, in this and the following paragraph, we restrict ourselves to nondegenerate conics.

† What tacit agreement is being made here as to the definition of the tangent to a conic at a point at infinity?

DEFINITION. *A tangent to a curve at a point at infinity is an asymptote of the curve, provided that the tangent is not the line at infinity.*

A parabola has no asymptotes. An ellipse, whether or not it has a real trace, has two conjugate-imaginary asymptotes which, as can readily be shown, intersect in the center. The asymptotes of a circle are, then, the isotropic lines through the center.

EXERCISES

1. Show that

$$\Delta = \tfrac{1}{2} \begin{vmatrix} 2A & B & D \\ B & 2C & E \\ D & E & 2F \end{vmatrix}$$

2. Prove that the equation of the tangent to the curve (1) at the point (r_1, r_2, r_3) is

$$A\,r_1x_1 + \frac{B}{2}(r_2x_1 + r_1x_2) + C\,r_2x_2 + \frac{D}{2}(r_3x_1 + r_1x_3) + \frac{E}{2}(r_3x_2 + r_2x_3) + F\,r_3x_3 = 0.$$

First solve the corresponding problem in nonhomogeneous coordinates.*

3. Find the point at infinity on the parabola

$$x^2 - 4xy + 4y^2 + 2x + y - 6 = 0,$$

and hence determine the direction of the axis.

4. Show that the tangents to the hyperbola represented by the first of the equations (4), at its points at infinity, are given by the second of the equations (4).

5. Prove that the asymptotes of either of the ellipses represented by the first two equations in (2) have the equations (3).

6. Find the equations of the asymptotes of the hyperbola

$$x^2 - xy - 2y^2 - x + 5y - 3 = 0,$$

and hence obtain the coordinates of the center.

9. Isotropic Tangents. Foci and Directrices. *Parabola.* There is a unique tangent to the parabola,

$$y^2 = 2mx, \tag{1}$$

with a prescribed slope $\lambda(\neq 0)$, namely †

$$y = \lambda x + \frac{m}{2\lambda}\cdot$$

Hence there are two tangents which are isotropic lines, one of slope

* See *Analytic Geometry*, p. 188, Ex. 2.
† See *Analytic Geometry*, Ch. IX, § 6.

i and one of slope $-i$:

$$(2) \qquad y = ix + \frac{m}{2i}, \qquad y = -ix - \frac{m}{2i}.$$

In what point do these isotropic tangents intersect? Evidently, in $(m/2, 0)$, the focus!

What are their points of contact with the parabola? The points of the parabola which lie on the directrix! For, since they have the common intercept, $m/2$, on the axis of x, their points of contact have the common abscissa, $-m/2$, and so lie on the directrix $x = -m/2$.*

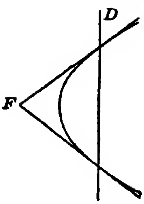


FIG. 3

THEOREM 1. *There are two isotropic tangents to a parabola. They intersect in the focus and their points of contact lie on the directrix.*

The relationships described are pictured *symbolically* in Fig. 3.

Ellipse. There are two tangents to the ellipse

$$(3) \qquad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \qquad a > b,$$

which have a given slope λ, namely

$$y = \lambda x \pm \sqrt{a^2\lambda^2 + b^2}.$$

There are, therefore, four isotropic tangents, two of each kind. Their equations are readily found to be

$$(4) \qquad \begin{array}{llll} (a) & y = i(x - c), & (c) & y = -i(x - c), \\ (b) & y = i(x + c), & (d) & y = -i(x + c). \end{array}$$

Two opposite vertices of the ordinary quadrilateral formed by the four isotropic tangents lie on the axis of x, in the real points $F : (c, 0)$ and $F' : (-c, 0)$, and the other two lie on the axis of y, in the conjugate-imaginary points $G : (0, ci)$ and $\bar{G} : (0, -ci)$. The quadrilateral is of the type discussed in § 5, Ex. 4. Its diagonals lie on the axes and bisect one another.

LEMMA. *If the intercepts of a tangent to* (3) *are A, B, the coordinates of the point of contact are* $(a^2/A, b^2/B)$.

* That the abscissa of the point of contact of a tangent to (1) is the negative of the x-intercept of the tangent is evident from the equation, $y_0 y = m(x + x_0)$, of the tangent at (x_0, y_0).

According to the Lemma, the points of contact of the two tangents through $F : (c, 0)$ lie on the line $D : x = a^2/c$ and those of the tangents through $F' : (-c, 0)$ lie on the line $D' : x = -a^2/c$. Similarly, the points of contact of the tangents through $G : (0, ci)$ are on the line $E : y = b^2/ci$ and those of the tangents through $G : (0, -ci)$ are on the line $\bar{E} : y = -b^2/ci$.

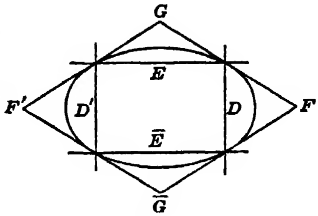


FIG. 4

The points F and F' are the usual foci of the ellipse, and the lines D and D' are the corresponding directrices. The points G and $\bar{G}$ we also agree to call foci and the lines E and $\bar{E}$, the corresponding directrices.

THEOREM 2. *An ellipse with a real trace has four isotropic tangents, two of each kind. The finite intersections of the isotropic tangents are the foci of the ellipse, and the points of contact of the two tangents through a given focus lie on the corresponding directrix.*

In the properties which an ellipse has with respect to the real foci and directrices, the transverse axis plays the principal rôle. There ought to be, and there actually are, similar properties with reference to the conjugate axis, and in these the conjugate-imaginary foci and directrices are fundamental.

The two sets of properties can best be exhibited in a table.

Foci	*Directrices*	*Properties*	
$F, F' : (\pm c, 0)$	$D, D' : x = \pm \dfrac{a^2}{c}$	$\pm FP \pm F'P = 2a,$	$\dfrac{FP^2}{PM^2} = e^2, \quad e^2 = \dfrac{c^2}{a^2}$
$G, \bar{G} : (0, \pm ic)$	$E, \bar{E} : y = \pm \dfrac{b^2}{ci}$	$\pm GP \pm \bar{G}P = 2b,$	$\dfrac{GP^2}{PN^2} = e'^2, e'^2 = -\dfrac{c^2}{b^2}$

The focus-directrix properties, as exhibited in the last two columns, are expressed in terms of the squares of the distances of a point P on the ellipse to a focus and to the corresponding directrix, in order to avoid the ambiguity which arises from the fact that distance in the complex plane is double-valued.

In the case of the properties appearing in the third column, this ambiguity cannot be avoided and the equations must be interpreted with care. The first equation, for example, really stands for the four

equations corresponding to the four possible combinations of signs (+, +), (−, −), (+, −), (−, +). It implies that FP and $F'P$, once they have been chosen as definite determinations of the distances from a point P on the ellipse to F and F', satisfy one of the four equations. Thus, if P is a real point and FP and $F'P$ are both taken as positive, it is $FP + F'P$ which is equal to $2a$.

EXERCISES

1. Verify the relations:

$$\pm GP \pm \bar{G}P = 2b, \qquad \left(\frac{GP}{PN}\right)^2 = -\frac{c^2}{b^2}.$$

Show that the sum of the squares of the reciprocals of the two eccentricities e, e' is unity.

2. Discuss the isotropic tangents, foci, and directrices of a hyperbola and summarize the results in a table.

3. The same for an ellipse without a real trace.

4. Determine the coordinates of the focus of the parabola

$$x^2 + 2xy + y^2 - 2x + 2y - 1 = 0.$$

5. Find the foci of the hyperbola

$$6x^2 - 24xy - y^2 - 150 = 0.$$

CHAPTER IX

ONE-DIMENSIONAL PROJECTIVE GEOMETRY

1. One-Dimensional Metric Coordinates. *Coordinates in a Range of Points.* As the basis for Cartesian coordinates on a real * line L, we choose on L a finite point O, a sense of direction, and a unit distance. The nonhomogeneous coordinate x of a finite point P is, then, the directed line-segment $\overline{OP}$:

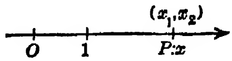


FIG. 1

$$x = \overline{OP}.$$

Corresponding homogeneous coordinates (x_1, x_2) of the finite point P consist of any two numbers x_1, x_2 whose ratio is x:

$$\frac{x_1}{x_2} = x, \qquad x_2 \neq 0.$$

When P recedes indefinitely on L in either direction, x becomes infinite, and conversely. Accordingly, we associate with the point at infinity on L the homogeneous coordinates $(1, 0)$ or, more generally $(\rho, 0)$, $\rho \neq 0$.

Every point on the *extended line* L now has homogeneous coordinates and every number pair (x_1, x_2), other than $(0, 0)$, serves as homogeneous coordinates of a point.

Coordinates in a Pencil of Lines. Let L_0 be a fixed line of the pencil of lines through a finite point and choose a positive sense for the measurement of angles at the point. A nonhomogeneous coordinate, u, of a line L of the pencil is the tangent of the angle θ from L_0 to L:

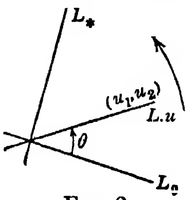


FIG. 2

$$u = \tan\theta.$$

The line L_* perpendicular to L_0 has no coordinate.

As homogeneous coordinates (u_1, u_2) of the line L, not L_*, we take any two numbers u_1, u_2 whose ratio is u:

$$\frac{u_1}{u_2} = u, \qquad u_2 \neq 0.$$

* We restrict ourselves for the time being to the domain of real points and real lines.

To L_* we give the homogeneous coordinates $(1, 0)$, or, more generally, $(\rho, 0)$, $\rho \neq 0$.

The Extended Line as a Closed Continuum. If the angle ϕ in Fig. 3 increases from 0 to 2π, the point P traces the circle. Thus the points of the circle are represented by the values of ϕ in the interval $0 \leqq \phi \leqq 2\pi$, where the point P_0 corresponds to both the values 0 and 2π.

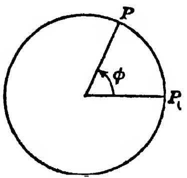


FIG. 3

The totality of real numbers in the interval $0 \leqq \phi \leqq 2\pi$ constitutes what is known as a *continuous* set of numbers, or a *continuum.* Accordingly, we call the totality of points on the circle a continuous set of points, or a *continuum.*

The points of the circle other than P_0, represented by the values of ϕ in the interval $0 < \phi < 2\pi$, also form a continuum. In this case we speak of an *open* continuum; in the previous case, of a *closed* continuum.

The lines of the pencil of Fig. 2 we can think of as represented by the values of θ in the interval $-\pi/2 \leqq \theta \leqq \pi/2$, where to L_* correspond both $-\pi/2$ and $\pi/2$. The totality of these lines forms a closed continuum.

If one line of the pencil is removed, there remains an open continuum. If L_* is removed, the open continuum which is left is represented by the values of θ in the interval $-\pi/2 < \theta < \pi/2$, or by all the values of the coordinate $u: -\infty < u < \infty$.

The points of the line of Fig. 1, exclusive of the point at infinity, are represented by the totality of values of the coordinate x: $-\infty < x < \infty$. They form an open continuum. When the point at infinity on the line is added, the continuum becomes closed.

In the cases of the circle and the pencil of lines, there is intuitive evidence both of the continuity and the closure. It is, however, impossible to visualize the extended straight line as having the intuitive properties of continuity and closure as well as that of straightness.

Though the geometric evidence of continuity "across the point at infinity" is lacking, the analytic proof is immediate. Since the totality of values of x_2 in the interval $1 \geqq x_2 \geqq -1$ is continuous, the totality of number pairs $(1000, x_2)$ where $1 \geqq x_2 \geqq -1$ is continuous. Hence, the totality of points on the extended line which have these number pairs as coordinates is continuous. But these are

(Fig. 4) all the points of the line beginning with the point $x = 1000$, continuing to the right "through the point at infinity" and returning from the left to the point $x = -1000$.

The extended straight line is a closed continuum.

FIG. 4

The reader may feel more reconciled to this fact after he has studied Fig. 5. This figure depicts a correspondence between the points of an extended line L and those of a circle C which is *continuous* and *one-to-one without exception.* The correspondence is established by the projection of L on C from the point O. In particular, to the point at infinity on L corresponds the point V' in which the line through O parallel to L meets C, and to the point O corresponds the point T in which the tangent to C at O meets L.

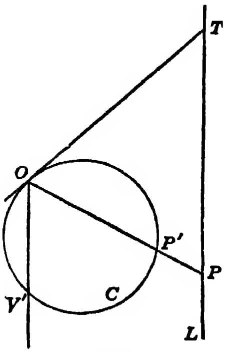


FIG. 5

2. Projective Coordinates in a Range of Points. We proceed to devise, for use in projective geometry, coordinates which are based only on projective properties.* We begin with the range of real points on a real line L.

DEFINITION. *If P_*, P_0, P_1 are three distinct fixed points, and P is an arbitrary point, of the range, the projective coordinate x of P, with respect to P_*, P_0, P_1, is defined as the cross ratio (P_*P_0, P_1P):* †

$$x = (P_*P_0, P_1P).$$

Each point P, other than P_*, has a unique coordinate x; in particular, P_0 has the coordinate 0, and P_1 the coordinate 1; see Ch. VI, § 7, Th. 1. The point P_* has no coordinate; when a variable point P approaches P_* as a limit, the coordinate x of P becomes infinite.

* Von Staudt (1798–1867) was the first to free projective geometry from all metric elements.

† Though the projective coordinate x is defined in terms of cross ratio, it is not, as here introduced, entirely divorced from metric geometry, inasmuch as we defined cross ratio originally in terms of distance. It is possible, however, starting from a projective definition of harmonic division (Ch. IV, § 7) to build up a theory of cross ratio and projective coordinates which is purely projective and in which a projective coordinate in a range of points appears, as here, as a cross ratio. A development along these lines is beyond the scope of the present book; see J. L. Coolidge, *Non-Euclidean Geometry*, Ch. XVIII.

The points P_*, P_0, P_1 are called the *basic points* of the coordinate system. Inasmuch as they can be chosen arbitrarily, there are infinitely many projective coordinate systems in the range.

THEOREM 1. *The metric coordinates on L are the special projective coordinates obtained by taking as P_* always the point at infinity on L.*

The metric coordinate X of an arbitrary point P on L, referred to a finite fixed point P_0 as origin and a directed line-segment $\overline{P_0P_1}$ as unit distance, is

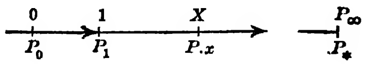


FIG. 6

$$X = \frac{\overline{P_0P}}{\overline{P_0P_1}}.$$

On the other hand, the projective coordinate x of P, with respect to P_∞ (as P_*), P_0, P_1, is

$$x = (P_\infty P_0,\ P_1P) = (PP_1,\ P_0P_\infty) = \frac{\overline{P_0P}}{\overline{P_0P_1}}.$$

Consequently, $X = x$: the metric coordinate of P is equal to the cross ratio in which the point at infinity and the origin are divided by the "unit-point," $X = 1$, and P.

Change of Coordinates. Let there be given on a line L a metric coordinate x and a projective coordinate x', both chosen at pleasure (Fig. 7). By definition

$$x' = (P_*P_0,\ P_1P).$$

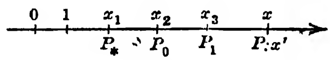


FIG. 7

But

$$(P_*P_0,\ P_1P) = \frac{(x_3 - x_1)(x - x_2)}{(x_3 - x_2)(x - x_1)},$$

where x_1, x_2, x_3, x are the metric coordinates of P_*, P_0, P_1, P. Hence

$$x' = \frac{(x_3 - x_1)(x - x_2)}{(x_3 - x_2)(x - x_1)}.$$

This equation represents the change from the metric coordinate x to the projective coordinate x'. Since x_1, x_2, x_3 are constants, it is of the form of a linear transformation

$$(1) \qquad x' = \frac{a_1x + a_2}{b_1x + b_2}, \qquad a_1b_2 - a_2b_1 \neq 0.$$

The inverse change, from x' to x, is also of the same form. Thus, the change from a metric to a projective coordinate, or vice versa, is given by a linear transformation.

Let x and x' now be any two projective coordinates on L and let $\bar{x}$ be a metric coordinate. The change from x to $\bar{x}$ and that from $\bar{x}$ to x' are given, as we have just proved, by linear transformations. Hence so is the change from x to x'; for the product of two linear transformations is a linear transformation.

THEOREM 2. *The change from one projective coordinate to a second is represented analytically by a linear transformation.*

Properties Expressed in Projective Coordinates. Let Q_1, Q_2, Q_3, Q_4 be four points on L with projective coordinates x_1', x_2', x_3', x_4' and metric coordinates x_1, x_2, x_3, x_4. The cross ratio $(Q_1Q_2,\ Q_3Q_4)$ is given, in terms of the metric coordinates, by the formula,

$$(Q_1Q_2,\ Q_3Q_4) = \frac{(x_3 - x_1)(x_4 - x_2)}{(x_3 - x_2)(x_4 - x_1)}. \tag{2}$$

The change from the metric coordinate x to the projective coordinate x' is given by a linear transformation and with respect to every linear transformation the expression on the right in (2) is an invariant. Hence

$$\frac{(x_3' - x_1')(x_4' - x_2')}{(x_3' - x_2')(x_4' - x_1')} = \frac{(x_3 - x_1)(x_4 - x_2)}{(x_3 - x_2)(x_4 - x_1)},$$

and therefore the value of the cross ratio in terms of the projective coordinates is

$$(Q_1Q_2,\ Q_3Q_4) = \frac{(x_3' - x_1')(x_4' - x_2')}{(x_3' - x_2')(x_4' - x_1')}. \tag{3}$$

THEOREM 3. *The expression for the cross ratio of four points on L is the same in projective as in metric coordinates.*

It follows that the analytic theory of projective transformations of the points of a line L into those of a line L' remains the same as before; for this theory depended solely on cross ratio.

THEOREM 4. *A projective transformation of the points of a line L into those of a line L', expressed in terms of projective coordinates on L and L', is a linear transformation of these coordinates, and conversely.*

Two Interpretations of Linear Transformations. In metric geometry, the equations

$$x' = x\cos\theta - y\sin\theta + a, \qquad y' = x\sin\theta + y\cos\theta + b,$$

are capable of two interpretations. We can think of them as representing either a rigid motion of the plane or a transformation from

the Cartesian coordinates (x, y) to new Cartesian coordinates (x', y'). In the first case, the coordinate system remains fixed and the points of the plane are shifted; in the second case, the points of the plane are fixed and new coordinates are assigned to them.

The linear transformation (1) lends itself, in our present work, to two analogous interpretations. It can be thought of as representing either a projective transformation of the line L into itself or a transformation from one system of projective coordinates on L to a second.

We have not, however, completely established this latter fact. We have proved merely that every transformation of projective coordinates is linear (Th. 2). It remains to show that the converse is true.

Let x be a projective coordinate on L and let the linear transformation (1) carry x into x'. To prove that x' is a projective coordinate on L.

Since (1) is uniquely determined by three distinct corresponding pairs of values, $x_1 \rightarrow x_1'$, $x_2 \rightarrow x_2'$, $x_3 \rightarrow x_3'$, an equation equivalent to (1) is

$$\frac{(x_3' - x_1')(x' - x_2')}{(x_3' - x_2')(x' - x_1')} = \frac{(x_3 - x_1)(x - x_2)}{(x_3 - x_2)(x - x_1)} .$$

In particular, let $x_1' = \infty$, $x_2' = 0$, $x_3' = 1$, that is, let x_1 be the value of x for which x' becomes infinite, and let x_2 and x_3 be the values of x for which x' has the values 0 and 1, respectively. Since $x_2' = 0$, $x_3' = 1$, our equation becomes

$$\frac{1 - x_1'}{x' - x_1'} \cdot x' = \frac{(x_3 - x_1)(x - x_2)}{(x_3 - x_2)(x - x_1)} .$$

When x_1' becomes infinite, the left-hand side has the limit x':

$$x' = \frac{(x_3 - x_1)(x - x_2)}{(x_3 - x_2)(x - x_1)} .$$

Hence x' is, by definition, the projective coordinate on L which is based on the points whose coordinates x are respectively x_1, x_2, x_3.

The Nature of the Projective Line. In establishing a system of projective coordinates, the point P_*, the point in which the coordinate becomes infinite, can be chosen at pleasure. On the other hand, the point which, in our intuitive conception of the line, we called the point at infinity can be made to take on any desired coordinate, by proper choice of the basic points. The point at infinity has, therefore, lost

its identity; it no longer differs from the other points of the line. Hence we conclude, in connection with the considerations of § 1:

A projective line is a closed continuum with no exceptional point. An extended metric line is a closed continuum with one exceptional point.

Homogeneous projective coordinates of the point P with the projective coordinate x are any two numbers (x_1, x_2) whose ratio x_1/x_2 equals x; those of P_* are $(1, 0)$ or $(\rho, 0)$, $\rho \neq 0$.

EXERCISES

1. The linear transformation,

$$\rho x_1' = 2x_1 - 4x_2, \qquad \rho x_2' = x_1 - x_2,$$

represents a change of homogeneous coordinates. Find the coordinates in each system of the basic points of the other system.

2. Find the linear transformation which represents the change from a given projective coordinate x to a new projective coordinate x' whose basic points P_*', P_0', P_1' are respectively $x = 3$, $x = -2$, $x = 5$.

3. Projective Coordinates in a Pencil of Lines. The treatment here is essentially the dual of that developed in § 2. The projective coordinate u of an arbitrary line L of a pencil with reference to three distinct basic lines L_*, L_0, L_1 is defined by the cross-ratio:

$$u = (L_*L_0, L_1L).$$

Corresponding homogeneous coordinates of L are (u_1, u_2), where $u_1/u_2 = u$, while those of L_* are $(1, 0)$ or $(\rho, 0)$, $\rho \neq 0$.

The change from a metric to a projective coordinate in the given pencil, and hence the change from one projective coordinate to a second, is given by a linear transformation. It follows that cross ratio in the pencil and projective transformations of the pencil into itself or into a second pencil have the same analytic representations in projective as in metric coordinates.

A Familiar Example of Projective Coordinates. If $a : (a_1, a_2, a_3)$ and $b : (b_1, b_2, b_3)$ are two distinct points, or lines, in the plane, every element, except a, of the range of points, or pencil of lines, determined by a and b has coordinates of the form $\mu a + b$. Since there is a one-to-one correspondence between the elements, other than a, of the range, or pencil, and the values of μ, μ is a coordinate in the range, or pencil. In fact, μ is a projective coordinate; for, the cross ratio of the four elements $\mu = \infty$, $\mu = 0$, $\mu = 1$, $\mu = \mu$, that is, of the four

elements a, b, $a + b$, $\mu a + b$, is precisely μ. Similarly, if $ka + lb$ is used instead of $\mu a + b$, (k, l) are homogeneous projective coordinates in the range, or pencil. Hence, we have long since been employing projective coordinates in both ranges of points and pencils of lines.

EXERCISES

1. Show geometrically that a metric coordinate in a pencil of lines is a special projective coordinate.

2. State and prove the duals of Ths. 2, 3, 4 of § 2.

4. Projective Correspondences. By means of projective coordinates we can give a simpler treatment of projective correspondences and transformations. At the same time we shall enlarge the scope of our theory.

In Ch. VII we considered in detail projective correspondences between two ranges of points. Similar correspondences can also be established between two pencils of lines and between a range of points and a pencil of lines. To avoid the awkwardness of separate statements in the three cases, we agree to call ranges of points and pencils of lines by a single name: *one-dimensional fundamental forms.* The definitions of a projective correspondence in the three cases can then be written as one.

DEFINITION. *Two one-dimensional fundamental forms are in projective correspondence if the elements of one are in one-to-one correspondence with the elements of the other so that corresponding cross ratios are equal.*

Similarly, in the case of the fundamental theorem:

THEOREM 1. *There exists a unique projective correspondence between two one-dimensional fundamental forms which orders to three given distinct elements of the one form three prescribed distinct elements of the other.*

We choose to give the proof in the case of two pencils of lines. Take the three given lines of the one pencil as the basic lines L_*, L_0, L_1 for a projective coordinate u, and the three corresponding lines of the other pencil as the basic lines L'_*, L'_0, L'_1 for a projective coordinate u'. A projective correspondence which orders L'_* to L_*, L'_0 to L_0, and L'_1 to L_1 must order to a line $L : u$ of the first pencil the line $L' : u'$ of the second for which

$$(L'_*L'_0,\ L'_1L') = (L_*L_0,\ L_1L).$$

Hence

$$u' = u. \tag{1}$$

But the correspondence established by this equation is surely projective. Hence, the theorem is proved.

The projective transformation (1) of the first pencil into the second is a linear transformation. Moreover, it remains linear if, instead of u and u', other projective coordinates are introduced, since the change from one projective coordinate to another is given by a linear transformation. Conversely, every linear transformation of a projective coordinate u of the first pencil into a projective coordinate u' of the second has the characteristic properties of a projective transformation.

THEOREM 2. *The projective transformations of a one-dimensional form into a second one-dimensional form are identical with the linear transformations of a projective coordinate in the first form into a projective coordinate in the second.*

5. Complex Projective Geometry. Consider, in connection with a projective coordinate system in a one-dimensional form, the arbitrary complex number, $w = w' + i\,w''$. If w is real, it is the coordinate of an element already present, a *real* element. If w is imaginary, there is created a new element called an *imaginary* element, to which w is given as a coordinate. The totality of real and imaginary elements constitutes the *complex* one-dimensional form.

To define the cross ratio of four complex elements of the form, we agree to take the usual expression in their coordinates:

$$(E_1E_2,\ E_3E_4) = \frac{(w_3 - w_1)(w_4 - w_2)}{(w_3 - w_2)(w_4 - w_1)}, \tag{1}$$

as long as it has a meaning. But the expression always has a meaning when the four elements are distinct, and, when they are not distinct, it gives rise to the same considerations and agreements as in the case of real elements (Ch. VI, § 7).*

Since the theory of cross ratio is the same for complex elements as for real elements, the theory of one-dimensional projective correspondences, so far as we have developed it, is the same for complex one-dimensional forms as for real one-dimensional forms.

* We note that, if E_*, E_0, E_1 are the basic elements of the form and E the element with coordinate w, then $w = (E_*E_0,\ E_1E)$ when the element E is imaginary as well as when it is real. This follows immediately from (1) when we let w_1 become infinite and replace w_2, w_3, w_4 by 0, 1, w respectively.

EXERCISES

1. Show that, if the ratio of an imaginary number to its conjugate is real, the ratio equals -1. Hence establish the theorem: The cross ratio in which two real elements are separated by two conjugate-imaginary elements is real if and only if the two pairs of elements form a harmonic set.

2. In connection with Ch. VI, § 5, Th. 5, prove that, if one of the twenty-four cross ratios of four distinct complex elements is an imaginary cube root of -1, that is, one of the numbers $\omega_1 = \frac{1}{2}(1 + \sqrt{3}\,i)$, $\omega_2 = \frac{1}{2}(1 - \sqrt{3}\,i)$, twelve of the cross ratios are equal to ω_1, the other twelve to ω_2. The four elements are said to form, in this case, an *equi-anharmonic* set.

6. Fixed Elements and Invariant of One-Dimensional Projective Transformations. Let (w_1, w_2) be homogeneous projective coordinates in a one-dimensional form, and consider a *real* projective transformation of the form into itself,

$$(1)\qquad \begin{aligned} \rho w_1' &= a_1 w_1 + a_2 w_2, \\ \rho w_2' &= b_1 w_1 + b_2 w_2, \end{aligned} \qquad \Delta = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \neq 0,$$

where a_1, a_2, b_1, b_2 are real constants.

An element is a *fixed*, or *double*, element of the transformation (1) if it is carried by (1) into itself. The element (w_1, w_2) is a fixed element if and only if

$$\frac{w_1}{w_2} = \frac{a_1 w_1 + a_2 w_2}{b_1 w_1 + b_2 w_2},$$

or

$$(2)\qquad b_1 w_1^2 + (b_2 - a_1) w_1 w_2 - a_2 w_2^2 = 0.$$

If $b_1 = b_2 - a_1 = a_2 = 0$, (1) is the identity and every element is fixed. Otherwise equation (2) guarantees just two fixed elements, which are real and distinct, real and coincident, or conjugate-imaginary, according as the discriminant of (2),

$$D = (b_2 - a_1)^2 + 4\,a_2 b_1 = (a_1 + b_2)^2 - 4\,\Delta,$$

is positive, zero, or negative.

THEOREM 1. *A linear transformation, other than the identity, has two fixed elements.*

According as the fixed elements are real and distinct, real and coincident, or conjugate-imaginary, the transformation is known as a *hyperbolic*, *parabolic*, or *elliptic* transformation.

*Nonparabolic Transformations.** Of the three distinct pairs of corresponding elements necessary and sufficient to determine uniquely

* The nonparabolic transformations comprise the elliptic and hyperbolic transformations, but not the identity.

a linear transformation, two pairs can be taken, in the case of a non-parabolic transformation, as the self-corresponding double elements. Hence:

THEOREM 2. *A nonparabolic linear transformation is uniquely determined by its double elements and a pair of corresponding elements.*

If E_1 and E_2 are the double elements, E_3 and E_3' the given pair, and E, E' an arbitrary pair, of corresponding elements, the equation of the transformation in symbolic form is

$$(E_1E_2,\ E_3'E') = (E_1E_2,\ E_3E).$$

Writing out the equation in full in terms of the *nonhomogeneous* coordinates of the various elements,—w_1 and w_2 for E_1 and E_2, w_3 and w_3' for E_3 and E_3', and w and w' for E and E'—and transposing factors so that the coordinates of E and E' appear only on the left-hand side, while those of E_3 and E_3' appear only on the right-hand side, we find

$$(E_1E_2,\ EE') = (E_1E_2,\ E_3E_3').$$

The right-hand side of this equation is constant, equal, say, to k, where $k \neq 0, 1$. The equation expresses, then, the important fact:

THEOREM 3. *The cross ratio in which an arbitrary pair of corresponding elements separates the double elements, taken in a given order,* is constant:*

$$(3) \qquad (E_1E_2,\ EE') = k, \qquad k \neq 0, 1.$$

This cross ratio is known as the *invariant cross ratio*, or simply the *invariant*, of the nonparabolic transformation, inasmuch as it actually is invariant with respect to a change of the projective coordinates in the one-dimensional form.

The transformation is now expressed by the symbolic equation (3). Writing out the cross ratio on the left-hand side of (3) and transposing the factors containing the coordinate of E, we obtain as the actual equation

$$(4) \qquad \frac{w' - w_2}{w' - w_1} = k\frac{w - w_2}{w - w_1}, \qquad k \neq 0, 1,$$

where w_1 and w_2 are the coordinates of the fixed elements and k is the invariant.

* There are really two cross ratios, according to which double element is taken as E_1 and which as E_2. The two are reciprocals of one another. We pick one and abide by the choice.

THEOREM 4. *A nonparabolic linear transformation is uniquely determined by its double elements, in a given order, and its invariant.*

If the transformation is *hyperbolic*, the double elements are real and the invariant k is real. By a change of projective coordinates, the double elements can be made to assume the homogeneous coordinates (1, 0) and (0, 1). The new equation of the transformation, which can be obtained directly from (4) by setting $w_2 = 0$ and letting w_1 become infinite, has the simple form

$$w' = kw, \qquad k \neq 0, 1. \tag{5}$$

Thus, by proper choice of the coordinate system, the equation of any hyperbolic transformation can be reduced to the *normal form* (5).

In the case of an *elliptic* transformation, the double elements are conjugate-imaginary and can be given the nonhomogeneous coordinates i and $-i$ by a change of coordinates. We thus obtain, as a *normal form* for an elliptic transformation,

$$\frac{w'+i}{w'-i} = k\frac{w+i}{w-i}, \qquad k \neq 0, 1, \tag{6 a}$$

or

$$(1-k)(ww'+1) + i(1+k)(w-w') = 0. \tag{6 b}$$

According to § 5, Ex. 1, the invariant k is of the form $m/\bar{m}$, where m is an imaginary number. The only real value which k can have is -1.

Parabolic Transformations. If, by a change of coordinates, the double-counting fixed element of a parabolic transformation has been given the homogeneous coordinates (1, 0), equation (2) must yield (1, 0) as the only fixed element. Hence, $b_1 = 0$, $b_2 - a_1 = 0$. We thus obtain, as a normal form of a parabolic transformation,

$$w' = w + c, \qquad c \neq 0. \tag{7}$$

EXERCISES

1. Find directly, without the use of the formulas in the text, the fixed elements and invariant of each of the transformations:

$$(a)\quad w' = \frac{4w-2}{w+1}; \qquad (b)\quad w' = -\frac{w+5}{w+1}.$$

2. Find the equation of the nonparabolic transformation whose double elements are $w_1 = 1$, $w_2 = -1$ and whose invariant k has the value 2.

3. The same, if $w_1 = 1 + i$, $w_2 = 1 - i$ and $k = i$.

4. The same, if $w_1 = -1$, $w_2 = 3$ and the transformation carries $w = 1$ into $w' = 0$.

5. Show that, if

$$w' = \frac{a_1 w + a_2}{b_1 w + b_2} \tag{8}$$

is a nonparabolic transformation and w_1, w_2 are its fixed elements, the invariant k has the value

$$k = \frac{a_1 w_1 + b_2 w_2}{b_2 w_1 + a_1 w_2},$$

provided the right-hand side has meaning. (Suggestion. Compare the coefficients in (4), when rewritten in the form (8), with the coefficients in (8).) Hence, show that always

$$k = \frac{a_1 + b_2 \pm \sqrt{(a_1 + b_2)^2 - 4\Delta}}{a_1 + b_2 \mp \sqrt{(a_1 + b_2)^2 - 4\Delta}}, \tag{9}$$

where the choice of signs depends on which fixed element is taken as w_1 and which as w_2.

6. Prove that the double element of a parabolic transformation and an arbitrary element E are separated harmonically by the element into which E is carried and the element which is carried into E.

7. Show that, if a double element of a linear transformation and a single element E have the property described in Ex. 6, the transformation is parabolic.

7. Involutions. The general nonparabolic linear transformation with invariant -1 has as its equation

$$(E_1E_2, EE') = -1, \tag{1}$$

where E_1, E_2 are the double elements. The transformation evidently carries each element into its harmonic conjugate with respect to the double elements. Hence, if it carries E into E', it also carries E' into E: if $(E_1E_2, EE') = -1$, then $(E_1E_2, E'E) = -1$. In other words, *the transformation interchanges the elements of each pair of elements which separate the double elements harmonically.*

Since the transformation, when it carries E into E', also carries E' into E, it is its own inverse. A transformation which has this property is known as an involutory transformation.

DEFINITION. *A transformation T, other than an identical transformation, is involutory if and only if it is its own inverse.*

The product of a transformation T and its inverse is the identical transformation: $TT^{-1} = I$. Hence, if T is its own inverse, the product of T with itself is the identity: $TT = I$. Conversely, if

$TT = I$, then T has the effect of undoing its own work and hence is its own inverse.

THEOREM 1. *A necessary and sufficient condition that a transformation, other than an identical transformation, be involutory is that the product of it with itself be the identity.*

We have noted that the linear transformations which are nonparabolic with invariant -1 are involutory. There are no other involutory linear transformations.

THEOREM 2. *A projective transformation of a one-dimensional form into itself is involutory if and only if it is nonparabolic and has -1 as its invariant.*

In establishing the theorem, we consider first a nonparabolic involutory transformation. Let its invariant be k and its double elements, E_1, E_2. When it carries E into E', it also carries E' into E:

$$(E_1E_2, EE') = k, \qquad (E_1E_2, E'E) = k, \qquad k \neq 0, 1.$$

The second cross ratio is the reciprocal of the first and therefore $k = 1/k$, or $k^2 = 1$. Consequently, since $k \neq 1$, $k = -1$.

It remains to prove that a parabolic transformation is never involutory. This is evident from the equation of the transformation written in the normal form, $w' = w + c$, $c \neq 0$, established in § 6.

An involutory projective transformation is commonly known as an *involution*. Employing this term, we summarize our results as follows.

THEOREM 3. *An involution in a one-dimensional form is a nonparabolic linear transformation with invariant -1. It amounts simply to an interchange of the elements of every pair of elements which separate harmonically two distinct fixed elements.*

The pairs of elements are called *pairs in the involution* and their totality is said to constitute the involution.

THEOREM 4. *An involution is uniquely determined by its fixed elements.*

If the fixed elements are real, the involution is *hyperbolic*. A normal form in this case, obtained from equation (5) of § 6 by setting $k = -1$, is

$$w' = -w. \tag{2}$$

If the fixed elements are conjugate-imaginary, the involution is

elliptic. Setting $k = -1$ in equation ($6b$) of § 6, we obtain for this type of involution the normal form

$$ww' + 1 = 0. \tag{3}$$

THEOREM 5. *Each two distinct pairs of real elements in an involution do not separate one another or do separate one another, according as the involution is hyperbolic or elliptic.*

Two distinct pairs of real elements E, E' and $\bar{E}$, $\bar{E}'$ do not separate or do separate one another according as $(EE', \bar{E}\,\bar{E}')$ is positive or negative. For the typical hyperbolic and elliptic involutions, (2) and (3), this cross ratio reduces respectively to

$$\left(\frac{\bar{w} - w}{\bar{w} + w}\right)^2 > 0, \qquad -\left(\frac{\bar{w} - w}{w\bar{w} + 1}\right)^2 < 0,$$

and hence the theorem is proved.

THEOREM 6. *If a linear transformation T interchanges the elements of just one pair of distinct elements, it is an involution.*

Suppose that T interchanges the elements A and A', to prove that always, when T carries E into E', it carries E' into E; that is, to show that if T carries E into E' and E' into $\bar{E}$, then $\bar{E}$ is always identical with E. Since A, A', E, E' go into A', A, E', $\bar{E}$,

$$(AA', EE') = (A'A, E'\bar{E}).$$

Hence

$$(AA', EE') = (AA', \bar{E}E'),$$

and $\bar{E}$ is identical with E.

THEOREM 7. *There is a unique involution which interchanges the elements of each of two given pairs of distinct elements.*

Let A, A' and B, B' be the given pairs. We know that there is a unique linear transformation which carries A into A', A' into A, and B into B', and this linear transformation is, by Th. 6, an involution.

Theorem 7 leads to the important conclusion:

THEOREM 8. *There is a unique pair of elements which separates harmonically each of two given pairs of distinct elements.*

The required pair consists of the double elements of the involution determined by the two given pairs, whose elements we now assume to be real. Its elements are real or conjugate-imaginary according as the involution is hyperbolic or elliptic. Hence, by Th. 5:

COROLLARY. *The elements are real or conjugate-imaginary according as the two given pairs do not separate or do separate one another.*

EXERCISES

1. Find the equation of the involution whose double elements have the nonhomogeneous coordinates $w = 1$, $w = 4$.

2. The same if the coordinates of the double elements are $w = 1 + i$, $w = 1 - i$.

3. Show that the transformation $w' = (4 - 3w)/(2w + 3)$ is an involution.

4. Find the equation of the involution which interchanges the elements $w = 1$, $w = 2$, and the elements $w = 0$, $w = 4$.

5. Determine the coordinates of the elements which separate harmonically each of the pairs of elements of Ex. 4.

6. The same if the two given pairs of elements are $w = 1$, $w = 4$ and $w = 0$, $w = 2$.

7. Describe completely the involutions represented by the equations (2) and (3) when w is thought of as a metric coordinate (slope) in a pencil of lines with finite vertex.

8. The same for the involution defined by (3) when w is interpreted as a metric coordinate on a finite line.

9. Prove that the linear transformation (8) of § 6 is an involution if and only if $a_1 + b_2 = 0$.

8. Metric Involutions. *In a Pencil of Lines with Finite Vertex.* The most important involution in a pencil of lines consists of the pairs of perpendicular lines in the pencil. We shall call it the *circular* involution in the pencil. Its double lines are the isotropic lines of the pencil and its equation, in terms of a metric coordinate (slope) in the pencil, is $u'u + 1 = 0$.

THEOREM 1. *An involution in a pencil, other than the circular involution, contains just one pair of perpendicular lines.*

It contains one pair of perpendicular lines, the bisectors of the angles formed by the double lines, and if it contained two or more, it would have at least two pairs of lines in common with the circular involution and hence, by § 7, Th. 7, would be identical with the circular involution.

In a Range of Points on a Finite Line. We consider first the general case in which the point at infinity is not a double point and hence is paired with a finite point. This finite point is called the *center* of the involution. The equation of the involution, expressed in terms of a metric coordinate x on the line, is of the form (§ 7, Ex. 9):

$$x' = \frac{a_1 x + a_2}{b_1 x - a_1}.$$

If x is in particular referred to the center as origin, the limit of x' when x becomes infinite must be zero, and hence $a_1 = 0$. Thus we obtain, as a normal form for the equation of the involution,

$$x'x = c, \qquad c \neq 0.$$

But this equation says:

THEOREM 2. *The product of the directed distances from the center of an involution on a line to a pair of points in the involution is constant.*

In the special case in which the point at infinity is a double point, it is geometrically evident that *the points of each pair in the involution are equally distant from the finite double point.*

An Application. Let a pencil of circles through the distinct points A and B be given (Fig. 8), and let L be a line which cuts the common chord AB in a finite point O external to the segment AB. Then, *the pairs of points in which L cuts the circles of the pencil form a hyperbolic involution with O as center.* For, if P and P' are the points of intersection of an arbitrary circle C of the pencil with L,

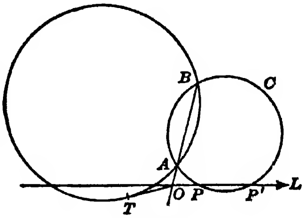


FIG. 8

$$\overline{OP} \cdot \overline{OP'} = OA \cdot OB,$$

and hence, since $OA \cdot OB$ is constant, the coordinates x and x' of P and P', referred to O as origin, are connected by the relation

$$x\,x' = c^2, \qquad c^2 = OA \cdot OB.$$

From the figure we see that $OT^2 = OA \cdot OB = c^2$. Hence a circle with O as center and OT as radius will intersect L in the points $x = \pm c$, that is, in the double points of the involution. It is evident that this fact can be used to effect a construction of the double points of a (hyperbolic) involution which is determined by two pairs of real points not separating one another.

EXERCISES

1. Two perpendicular lines which are paired in an involution in a pencil of lines may be called the *principal* lines of the involution. Show that, when the slope u in the pencil is referred to a principal line as initial line ($u = 0$), the equation of the involution reduces to the normal form $u'u = c$, $c \neq 0$. What geometric theorem results?

2. The double lines of an involution in the pencil of lines through the origin in the metric plane are $x^2/A + y^2/B = 0$. Show that the equation of the involution, in terms of the slope λ, is $\lambda\lambda' = -B/A$.

3. Prove that, if P_1, P_2 and Q_1, Q_2 form a harmonic set, the mid-points of the segments P_1P_2, Q_1Q_2 are paired in the involution determined by P_1, P_2 and Q_1, Q_2.

4. Give in detail the construction, referred to in the text, of the pair of points separating harmonically two given pairs of points which do not separate one another.

5. Prove that through two points A, B and tangent to a line L meeting the line AB in a point external to the segment AB there are two circles. Give a construction for them.

6. What does the theorem of Fig. 8 become when L is chosen so that O is internal to the segment AB? Verify your answer.

EXERCISES ON CHAPTER IX

1. Show that an involution contains infinitely many or no pairs of conjugate-imaginary elements, according as it is hyperbolic or elliptic. Hence prove that two conjugate-imaginary elements can never separate two other conjugate-imaginary elements harmonically.

2. Prove that two involutions in the same one-dimensional form whose double elements are distinct have just one pair of elements in common, and that the elements of this pair are either real or conjugate-imaginary. By Ex. 1 prove that the elements are real if at least one of the involutions is elliptic. What are the facts if neither is elliptic?

3. Show that a line which does not contain a vertex of a complete quadrangle cuts the pairs of opposite sides of the quadrangle in pairs of points in an involution.

4. Describe a construction, based on the theorem of Ex. 3, for the point corresponding to an arbitrarily chosen point in an involution which is determined by two given pairs of distinct points.

Bilinear Forms. The polynomial on the left-hand side of the equation

$$(1\,a) \qquad c_1w_1w_1' + c_2w_1w_2' + c_3w_2w_1' + c_4w_2w_2' = 0, \qquad c_1c_4 - c_2c_3 \neq 0.$$

is linear and homogeneous in w_1, w_2 and also linear and homogeneous in w_1', w_2'. It is known as a nonsingular *bilinear form* in the two sets of homogeneous variables w_1, w_2 and w_1', w_2'. The corresponding equation in the nonhomogeneous variables w and w' is

$$(1\,b) \qquad c_1ww' + c_2w + c_3w' + c_4 = 0.$$

5. Show that $(1\,a)$ or $(1\,b)$ represents the general projective transformation of a one-dimensional fundamental form into itself.

6. Prove that this projective transformation is an involution if and only if $c_2 = c_3$, that is, that the general involution is represented by

$$c_1w_1w_1' + c_2(w_1w_2' + w_2w_1') + c_4w_2w_2' = 0, \qquad c_1c_4 - c_2^2 \neq 0,$$

or

$$c_1ww' + c_2(w + w') + c_4 = 0.$$

7. Show that the pairs of distinct elements (r_1, r_2), (r_1', r_2'), (s_1, s_2), (s_1', s_2'), (t_1, t_2), (t_1', t_2') are pairs in an involution if and only if

$$\begin{vmatrix} r_1r_1' & r_1r_2' + r_2r_1' & r_2r_2' \\ s_1s_1' & s_1s_2' + s_2s_1' & s_2s_2' \\ t_1t_1' & t_1t_2' + t_2t_1' & t_2t_2' \end{vmatrix} = 0.$$

What is the form of the condition in nonhomogeneous coordinates?

8. Write an equation in determinant form for the involution determined by two pairs of distinct elements (r_1, r_2), (r_1', r_2'), (s_1, s_2), (s_1', s_2').

Quadratic Forms. The equation

$$a_{11}w_1^2 + 2\,a_{12}w_1w_2 + a_{22}w_2^2 = 0, \qquad a_{11}a_{22} - a_{12}^2 \neq 0, \tag{2}$$

represents a pair of distinct elements. If the coefficients are real, the elements are real or conjugate-imaginary.

9. Show that the elements (2) are separated harmonically by the distinct elements (r_1, r_2), (s_1, s_2) if and only if

$$a_{11}r_1s_1 + a_{12}(r_1s_2 + r_2s_1) + a_{22}r_2s_2 = 0. \tag{3}$$

Suggestion. Prove that the elements (2) can have their coordinates expressed in the forms $r + \lambda's$, $r + \lambda''s$, where λ', λ'' are the roots of a certain quadratic equation. The condition for harmonic division, $\lambda' + \lambda'' = 0$, will then be found equivalent to (3).

10. Show that the two pairs of elements, (2) and

$$b_{11}w_1^2 + 2\,b_{12}w_1w_2 + b_{22}w_2^2 = 0, \qquad b_{11}b_{22} - b_{12}^2 \neq 0, \tag{4}$$

form a harmonic set if and only if

$$a_{11}b_{22} - 2\,a_{12}b_{12} + a_{22}b_{11} = 0.$$

11. By means of Ex. 9 prove that the involution whose double elements are defined by (2) has the equation

$$a_{11}w_1w_1' + a_{12}(w_1w_2' + w_2w_1') + a_{22}w_2w_2' = 0. \tag{5}$$

12. Using Ex. 10, show that the pairs of elements in the involution determined by the two pairs of distinct elements (2) and (4) are given by

$$k\,(a_{11}w_1^2 + 2\,a_{12}w_1w_2 + a_{22}w_2^2) + l\,(b_{11}w_1^2 + 2\,b_{12}w_1w_2 + b_{22}w_2^2) = 0,$$

and that the double elements of the involution,—the pair of elements which separates harmonically each of the pairs of elements (2) and (4)—are defined by

$$\begin{vmatrix} a_{11}w_1 + a_{12}w_2 & a_{12}w_1 + a_{22}w_2 \\ b_{11}w_1 + b_{12}w_2 & b_{12}w_1 + b_{22}w_2 \end{vmatrix} = 0. \tag{6}$$

CHAPTER X

PROJECTIVE COORDINATES IN THE PLANE

1. Projective Point Coordinates. As basic points of a system of projective point coordinates in the real extended plane, take four real points, A_1, A_2, A_3, D, no three collinear. Denote the sides of the triangle $A_1A_2A_3$ by a_1, a_2, a_3, and the lines joining the vertices to D by d_1, d_2, d_3.

Choose arbitrarily a point P of the plane, draw the lines p_1, p_2, p_3 joining P to A_1, A_2, A_3, and form the cross ratios

$$(1)\quad \lambda_1 = (a_2a_3,\ d_1p_1), \qquad \lambda_2 = (a_3a_1,\ d_2p_2), \qquad \lambda_3 = (a_1a_2,\ d_3p_3).$$

If the cross ratios are all defined, their product, since p_1, p_2, p_3 are concurrent, is equal to unity (Ch. VI, § 6, Ex. 2):

$$(2)\qquad \lambda_1\lambda_2\lambda_3 = 1.$$

In general, if a point P is given, λ_1, λ_2, λ_3 are unique and satisfy (2); conversely, if λ_1, λ_2, λ_3 are given satisfying (2), p_1, p_2, p_3 are unique and concurrent, and hence determine a point P.

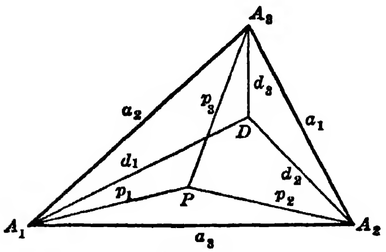


FIG. 1

Thus the cross ratios $\lambda_1, \lambda_2, \lambda_3$ constitute possible coordinates for P. But they have as coordinates two drawbacks. In the first place they must always satisfy (2). Secondly, if P lies on a side of the triangle $A_1A_2A_3$, one of them is undefined. For example, if P is a point on a_3, other than a vertex, p_1 and p_2 coincide with a_3 and

$$(3)\qquad \lambda_1 = (a_2a_3,\ d_1a_3) = 0, \qquad \lambda_2 = (a_3a_1,\ d_2a_3) = \infty.$$

Incidentally, in this and similar cases the relation (2) has no meaning.

Both objections can be met by replacing the λ's by suitably chosen ratios. Taking account of (2), we set

$$\lambda_1 = \frac{x_3}{x_2}, \qquad \lambda_2 = \frac{x_1}{x_3}, \qquad \lambda_3 = \frac{x_2}{x_1},$$

and introduce (x_1, x_2, x_3) as the coordinates of P.

Definition. *As the homogeneous projective coordinates of the point P we take any three numbers x_1, x_2, x_3 such that*

$$\frac{x_3}{x_2} = \lambda_1, \qquad \frac{x_1}{x_3} = \lambda_2, \qquad \frac{x_2}{x_1} = \lambda_3. \tag{4}$$

In justifying the definition we begin with points in special position. Let P be a vertex, say, A_1. Then p_1 is undefined, p_2 coincides with a_3, and p_3 with a_2. Hence λ_1 is undefined, and

$$\lambda_2 = (a_3a_1, d_2a_3) = \infty \quad \text{or} \quad \frac{1}{\lambda_2} = 0, \qquad \lambda_3 = (a_1a_2, d_3a_2) = 0.$$

Thus, the first equation in (4) is meaningless and the other two become

$$\frac{x_3}{x_1} = 0, \qquad \frac{x_2}{x_1} = 0.$$

Consequently, $x_1 \neq 0$, $x_2 = x_3 = 0$. Thus, A_1 has the coordinates $(\rho, 0, 0)$ or $(1, 0, 0)$. Similarly, A_2 and A_3 have respectively the coordinates $(0, 1, 0)$ and $(0, 0, 1)$.

Conversely, if $x_1 = \rho \neq 0$, $x_2 = x_3 = 0$, then $\lambda_2 = \infty$ and $\lambda_3 = 0$. Hence p_2 and p_3 coincide respectively with a_3 and a_2 and so intersect in A_1.

Let P now be any point on a_3, other than A_1 or A_2. According to (3), $\lambda_1 = 0$ and $1/\lambda_2 = 0$; hence equations (4) become

$$\frac{x_3}{x_2} = 0, \qquad \frac{x_3}{x_1} = 0, \qquad \frac{x_2}{x_1} = \lambda_3.$$

The first two equations say that $x_3 = 0$, $x_1x_2 \neq 0$; the third equation determines the ratio x_2/x_1, inasmuch as λ_3 is defined, $\neq 0$. Thus P has coordinates $(x_1, x_2, 0)$, where $x_1x_2 \neq 0$.

Let the reader prove, conversely, that a given number triple $(x_1, x_2, 0)$, where $x_1x_2 \neq 0$, is a set of coordinates of a point P on a_3, other than A_1 or A_2.

It follows now that a point P lies on a_3 if and only if its third coordinate is zero. The line a_3 has, therefore, the equation $x_3 = 0$. Similarly, a_1 has the equation $x_1 = 0$, and a_2, the equation $x_2 = 0$.

Consider, finally, a point P which is not on a side of the triangle. In this case, each λ is defined and not zero, and $\lambda_1\lambda_2\lambda_3 = 1$. Consequently, equations (4) have a solution, x_1, x_2, x_3, where $x_1x_2x_3 \neq 0$, and every other solution is of the form ρx_1, ρx_2, ρx_3, $\rho \neq 0$. For

example, if $\lambda_1 = 1/2$, $\lambda_2 = 1$, $\lambda_3 = 2$, one set of coordinates of P is $(1, 2, 1)$ and all sets are given by $(\rho, 2\rho, \rho)$, $\rho \neq 0$.

Conversely, three real numbers x_1, x_2, x_3, no one of which is zero, are the coordinates of a point not lying on a side of the triangle.

The justification of the definition is now complete. Incidentally, we have found that A_1, A_2, A_3 have respectively the coordinates $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$, and that a_1, a_2, a_3 have respectively the equations $x_1 = 0$, $x_2 = 0$, $x_3 = 0$.

The equations of d_1, d_2, d_3 are $x_2 = x_3$, $x_3 = x_1$, $x_1 = x_2$ respectively. Consider, for example, d_1. The coordinates $(1, 0, 0)$ of A_1 certainly satisfy $x_2 - x_3 = 0$, and a point $P : (x_1, x_2, x_3)$, other than A_1, lies on d_1 if and only if p_1 coincides with d_1, that is, if and only if $\lambda_1 = 1$ or $x_2 = x_3$. Hence $x_2 = x_3$ is the equation of d_1.

Since D lies on each of the lines d_1, d_2, d_3, its coordinates are all equal. Hence a simple set of coordinates for D is $(1, 1, 1)$. For this reason D is known as the *unit point* of the projective coordinate system. The triangle $A_1A_2A_3$ is called the *triangle of reference*.

THEOREM 1. *Cartesian coordinates are special projective coordinates.*

Let a system of homogeneous Cartesian coordinates (x_1, x_2, x_3) be given. Take, as the points A_1, A_2, A_3, D of the previous discussion, the points whose Cartesian coordinates are respectively $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$, $(1, 1, 1)$, that is, the points at infinity in the directions of the axes, the origin, and the point with nonhomogeneous coordinates $(1, 1)$. To verify our theorem we have to show that, if an arbitrary point P has the homogeneous Cartesian coordinates (r_1, r_2, r_3), then

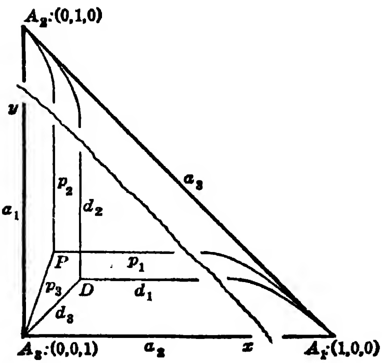


FIG. 2

$$\lambda_1 = \frac{r_3}{r_2}, \qquad \lambda_2 = \frac{r_1}{r_3}, \qquad \lambda_3 = \frac{r_2}{r_1},$$

where λ_1, λ_2, λ_3 are the three cross ratios (1).

The four lines a_2, a_3, d_1, p_1, whose cross ratio is λ_1, have the equations

$$x_2 = 0, \qquad x_3 = 0, \qquad x_2 - x_3 = 0, \qquad r_3x_2 - r_2x_3 = 0.$$

Hence $\lambda_1 = r_3/r_2$. Similarly, $\lambda_2 = r_1/r_3$ and $\lambda_3 = r_2/r_1$.*

The Nature of the Projective Plane. Since the basic points of a projective coordinate system are subject only to the restriction that no three be collinear, the line A_1A_2, that is, the line $x_3 = 0$, can be chosen as any line in the plane. Thus the so-called " line at infinity " loses its identity as the line $x_3 = 0$ and no longer differs from any other line.†

The projective plane has no exceptional points or lines. The extended metric plane has one exceptional line, the line at infinity, and, on this line, two exceptional points, the circular points at infinity.

The points of either plane constitute a closed two-dimensional continuum similar to that formed by the totality of lines through a point in space.

EXERCISES

1. What are the coordinates of the point in which the line d_1 meets the line a_1?

2. What is the equation of the line which is the harmonic conjugate of d_3 with respect to a_1 and a_2?

3. What does the equation $x_2 - 2x_3 = 0$ represent?

4. Prove that, if a point P, other than A_3, has the projective coordinates (r_1, r_2, r_3), the projection of P from A_3 on a_3 has coordinates $(r_1, r_2, 0)$.

5. Prove that $(x_1, x_2, 0)$ are homogeneous projective coordinates on the line a_3. What are the basic points?

2. Properties Expressed in Projective Coordinates. We first establish the equations of the transformation from an arbitrary Cartesian coordinate system to an arbitrary projective coordinate system. Let the Cartesian coordinates of the basic points A_1, A_2, A_3, D of the projective system be respectively (a_1, a_2, a_3), (b_1, b_2, b_3), (c_1, c_2, c_3), (d_1, d_2, d_3), and let (x_1, x_2, x_3) be the Cartesian, and (x_1', x_2', x_3') the projective, coordinates of an arbitrarily chosen point P.

To obtain the relationship between (x_1, x_2, x_3) and (x_1', x_2', x_3') we

* The proof covers merely the case of a point P which does not lie on a side of the triangle $A_1A_2A_3$.

† As a matter of fact, we have long since known that the line at infinity is projectively equivalent to every other line in that it can be transformed by a collineation of the plane into any prescribed line.

compute, by means of Ch. VI, § 4, Ex. 6, the values of the cross ratios λ_1, λ_2, λ_3 of § 1 in terms of the Cartesian coordinates of the various points. We find

(1)
$$\lambda_1 = \frac{\dfrac{|a\,b\,x|}{|a\,b\,d|}}{\dfrac{|c\,a\,x|}{|c\,a\,d|}}, \qquad \lambda_2 = \frac{\dfrac{|b\,c\,x|}{|b\,c\,d|}}{\dfrac{|a\,b\,x|}{|a\,b\,d|}}, \qquad \lambda_3 = \frac{\dfrac{|c\,a\,x|}{|c\,a\,d|}}{\dfrac{|b\,c\,x|}{|b\,c\,d|}}.$$

On the other hand,

$$\lambda_1 = \frac{x_3'}{x_2'}, \qquad \lambda_2 = \frac{x_1'}{x_3'}, \qquad \lambda_3 = \frac{x_2'}{x_1'}.$$

Consequently,

(2)
$$\rho x_1' = \frac{|b\,c\,x|}{|b\,c\,d|}, \qquad \rho x_2' = \frac{|c\,a\,x|}{|c\,a\,d|}, \qquad \rho x_3' = \frac{|a\,b\,x|}{|a\,b\,d|}, \quad \rho \neq 0.$$

These equations represent the transformation from the metric coordinates (x_1, x_2, x_3) to the projective coordinates (x_1', x_2', x_3'). Since the numerators are linear and homogeneous in x_1, x_2, x_3 and the denominators are constants, the transformation is linear. It is, in fact, the linear transformation which carries (a_1, a_2, a_3) into $(1, 0, 0)$, (b_1, b_2, b_3) into $(0, 1, 0)$, (c_1, c_2, c_3) into $(0, 0, 1)$, and (d_1, d_2, d_3) into $(1, 1, 1)$; see Ch. VII, § 10.

Since the transformation from metric coordinates to projective coordinates, or vice versa, is linear, the transformation from one projective coordinate system to another is linear.

Theorem 1. *The transformation from one projective coordinate system to a second is a linear transformation.*

We are now able to prove that the analytic representations of projective configurations and properties are the same in projective coordinates as in metric coordinates.

Consider an arbitrary line L. Let its equation in the Cartesian coordinates x_1, x_2, x_3 be

(3) $$a_1x_1 + a_2x_2 + a_3x_3 = 0, \quad a_1, a_2, a_3 \text{ not all zero.}$$

The equation of L in the projective coordinates x_1', x_2', x_3' is found by applying to (3) the transformation (2). Since (2) is a linear transformation, it carries the linear homogeneous equation (3) into a similar equation in x_1', x_2', x_3':

(4) $$a_1'x_1' + a_2'x_2' + a_3'x_3' = 0, \quad a_1', a_2', a_3' \text{ not all zero.}$$

Conversely, every equation of the form (4) is transformed by the inverse of (2) into an equation of the form (3).

THEOREM 2. *The equation of a straight line in projective coordinates is a linear homogeneous equation, and conversely.*

The theorem implies that the familiar analytic theory of points on a line reproduces itself in projective coordinates. From it we conclude that the three points with symbolic projective coordinates a, b, c are collinear if and only if $|a\,b\,c| = 0$, or if and only if they are linearly dependent. It follows that, if the points a and b are distinct, an arbitrary point of their range has coordinates $k\,a + l\,b$, k and l not both zero, and conversely.

Using the method employed to establish Theorem 2, we can prove that the expressions for the cross ratio of four collinear points are the same in projective coordinates as in metric coordinates. In particular:

THEOREM 3. *If P_1, P_2, P_3, P_4 are four distinct collinear points with the projective coordinates a, b, $a + \mu_1 b$, $a + \mu_2 b$, then*

$$(P_1P_2,\, P_3P_4) = \frac{\mu_1}{\mu_2}\,.$$

Since the cross ratio formula by means of which equations (1) were established is valid for projective coordinates, we can consider the x's as well as the x''s in equations (2) as projective coordinates. Then (2) is a transformation from one projective coordinate system to another.

If we can show that (2) is an arbitrary linear transformation, we shall have proved the converse of Theorem 1:

THEOREM 4. *Every linear transformation of x_1, x_2, x_3 into x_1', x_2', x_3' represents a change from one projective coordinate system to a second.*

Let an arbitrary linear transformation be given. It is uniquely determined by *any* four pairs of corresponding number triples, provided no three triples of either set of four are linearly dependent. Accordingly, we can think of it as the linear transformation which carries four *arbitrary* triples a, b, c, d, no three linearly dependent, respectively into $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$, $(1, 1, 1)$. This transformation is precisely the transformation (2). Hence, Theorem 4 is proved.

A linear transformation,

$$\rho x_i' = \sum_{j=1}^{3} a_{ij}x_j, \quad (i = 1, 2, 3), \quad |a_{ij}| \neq 0,$$

has now two interpretations. It can be thought of as a transformation from one projective coordinate system to another. Thereby the points of the plane remain fixed and their coordinates are changed.

On the other hand, a linear transformation represents a projective transformation of the plane into itself or into a second plane. We have, in fact:

THEOREM 5. *A projective transformation of the points of a plane M into the points of a plane M', expressed in projective coordinates in the two planes, is a linear transformation, and conversely.*

The proof is identical with that of Ch. VII, §§ 9, 11, for the properties on which the proof depends have been shown to have identically the same expressions in projective, as in metric, coordinates.

EXERCISES

1. The linear transformation,

$$\rho x_1' = -x_1 + x_2 + x_3, \qquad \rho x_2' = x_1 - x_2 + x_3, \qquad \rho x_3' = x_1 + x_2 - x_3,$$

represents a change of coordinates. Find the coordinates in each system of the basic points of the other system. What are the equations in the first system of the sides of the triangle of reference of the second system?

2. Find the equations of the most general transformation of projective coordinates which introduces the lines

$$2x_1 - 3x_2 + x_3 = 0, \qquad x_1 + 2x_2 - 3x_3 = 0, \qquad x_1 - x_2 + x_3 = 0$$

as the sides $x_1' = 0$, $x_2' = 0$, $x_3' = 0$ of the new triangle of reference. Then determine the particular transformation which also introduces (2, 3, 2) as the new unit point.

3. Find the change of projective coordinates which introduces the points (1, 0, 1), (0, 1, 1), (1, 2, 1), (2, 1, 1) as the new basic points A_1, A_2, A_3, D. Use the method suggested by the previous exercise and then check the result by means of equations (2) of the text.

4. Two projective coordinate systems have the same triangle of reference $A_1A_2A_3$, but different unit points. Determine the form of the transformation from one system to the other.

5. Four points which lie on a line not passing through A_3 have the symbolic coordinates a, b, c, d. Show that their cross ratio in the order given is equal to that of the four points $(a_1, a_2, 0)$, $(b_1, b_2, 0)$, $(c_1, c_2, 0)$, $(d_1, d_2, 0)$. Hence write the value of the cross ratio.

6. What is the degree of the equation in projective coordinates of a conic? Justify your answer.

7. Show that the equations,

$$\frac{|b'c'x'|}{|b'c'd'|} = \frac{|b\,c\,x|}{|b\,c\,d|}, \qquad \frac{|c'a'x'|}{|c'a'd'|} = \frac{|c\,a\,x|}{|c\,a\,d|}, \qquad \frac{|a'b'x'|}{|a'b'd'|} = \frac{|a\,b\,x|}{|a\,b\,d|},$$

represent the linear transformation which carries a, b, c, d respectively into a', b', c', d', where no three triples of either set of four are linearly dependent.

3. Projective Line Coordinates. Let a_1, a_2, a_3, d be four real lines in the plane, no three of which are concurrent. Denote the vertices of the triangle whose sides are a_1, a_2, a_3 by A_1, A_2, A_3, and the points of intersection of d with a_1, a_2, a_3 by D_1, D_2, D_3.

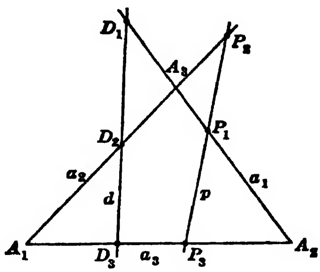


FIG. 3

Consider an arbitrary line p of the plane, mark its intersections P_1, P_2, P_3 with a_1, a_2, a_3, and form the cross ratios

$$\lambda_1 = (A_2A_3,\ D_1P_1), \qquad \lambda_2 = (A_3A_1,\ D_2P_2), \qquad \lambda_3 = (A_1A_2,\ D_3P_3).$$

Since the points P_1, P_2, P_3 are collinear,

$$\lambda_1\lambda_2\lambda_3 = 1,$$

provided each λ is defined.

DEFINITION. *Any three numbers u_1, u_2, u_3 such that*

$$\frac{u_3}{u_2} = \lambda_1, \qquad \frac{u_1}{u_3} = \lambda_2, \qquad \frac{u_2}{u_1} = \lambda_3$$

constitute homogeneous projective line coordinates of the line p.

The justification of the definition is left to the reader. He will find, in particular, that the lines a_1, a_2, a_3, and d have respectively the coordinates $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$, and $(1, 1, 1)$, and that the points A_1, A_2, A_3 have the equations $u_1 = 0,\ u_2 = 0,\ u_3 = 0$.

The line d is known as the *unit line*, and the triangle whose sides are a_1, a_2, a_3, as the *triangle of reference.*

If (u_1, u_2, u_3) and (u_1', u_2', u_3') are projective coordinates of the same line referred to two different systems of basic lines, the relationship between them is that of a linear transformation (Exs. 6, 7). All the properties constituting projective line geometry are invariant under linear transformations and therefore have the same analytic form in projective line coordinates as in metric line coordinates. A point is represented in projective line coordinates by a homogeneous linear equation, and conversely; the formulas for the cross ratio of four concurrent lines are preserved; and the projective transformations of the lines of a plane into the lines of the same, or a different, plane are

identical with the linear transformations of the projective line coordinates.

EXERCISES

1. Justify the definition of projective line coordinates.

2. What are the coordinates of the line A_3D_3?

3. What does the equation $2u_1 + u_2 = 0$ represent?

4. Show that the line which is determined by A_2 and the point of intersection with a_2 of a given line (u_1, u_2, u_3), not a_2, has the coordinates $(u_1, 0, u_3)$. Of what value is this fact?

5. Prove that $(u_1, 0, u_3)$ are homogeneous projective line coordinates in the pencil of lines whose vertex is A_2.

6. Show that metric line coordinates are special projective line coordinates.

7. Prove that the change from a metric system of line coordinates to a projective system, and hence the change from one projective system to a second, is given by a linear transformation.

8. The linear transformation,

$$\rho u_1' = a_1u_1, \qquad \rho u_2' = a_2u_2, \qquad \rho u_3' = a_3u_3, \qquad a_1a_2a_3 \neq 0,$$

represents a change from one system of reference for line coordinates to a second. How are the two systems of reference related?

4. Projective Point and Line Coordinates in the Same Plane. In developing simultaneously in the same plane a system of projective point coordinates and a system of projective line coordinates, there is one question of paramount importance: that of the analytic form of the condition that a point x lie on a line u. This condition must be linear and homogeneous in x_1, x_2, x_3, inasmuch as, when the u's are held fast and the x's varied, it represents a line. Similarly, it must be linear and homogeneous in u_1, u_2, u_3. Hence, it must be *bilinear* in x_1, x_2, x_3 and u_1, u_2, u_3; that is, it must be of the form

$$\begin{aligned} &a_{11}x_1u_1 + a_{12}x_1u_2 + a_{13}x_1u_3 \\ + &a_{21}x_2u_1 + a_{22}x_2u_2 + a_{23}x_2u_3 \\ + &a_{31}x_3u_1 + a_{32}x_3u_2 + a_{33}x_3u_3 = 0. \end{aligned} \tag{1}$$

The importance of the question at issue is obvious. The simplicity of the resulting analytic geometry depends largely on how simple (1) can be made by proper choice of the systems of reference for the point and line coordinates. It is to be hoped that these systems can be so chosen that (1) will become

$$u_1x_1 + u_2x_2 + u_3x_3 = 0. \tag{2}$$

Let us select at pleasure the triangle of reference $A_1A_2A_3$ and the unit point D for the system of point coordinates, and take the same triangle $a_1a_2a_3$ as the triangle of reference for the system of line coordinates (Fig. 4).

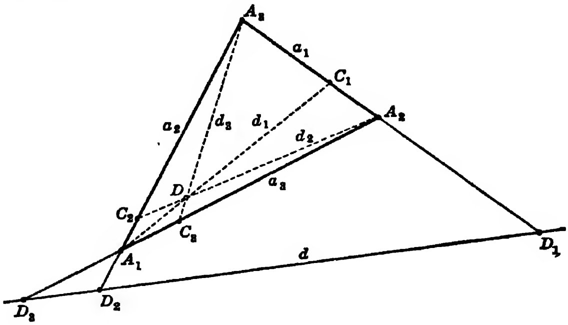


FIG. 4

The point $A_1 : (1, 0, 0)$ lies, then, on the line $a_2 : (0, 1, 0)$. Setting $x_1 = 1$, $x_2 = 0$, $x_3 = 0$ and $u_1 = 0$, $u_2 = 1$, $u_3 = 0$ in (1) must yield, therefore, a true equation. Thus we find that $a_{12} = 0$. Similarly, since $A_1 : (1, 0, 0)$ also lies on $a_3 : (0, 0, 1)$, we have $a_{13} = 0$. In the same way $a_{21} = 0$, $a_{23} = 0$, and $a_{31} = 0$, $a_{32} = 0$. Hence (1) becomes

$$a_{11}x_1u_1 + a_{22}x_2u_2 + a_{33}x_3u_3 = 0. \tag{3}$$

It remains to demand that the unit line $d : (1, 1, 1)$ be so chosen that (3) reduces to (2).

The equation of the unit line, as derived from (3), is

$$a_{11}x_1 + a_{22}x_2 + a_{33}x_3 = 0, \tag{4}$$

whereas that obtained from the desired form (2) is

$$x_1 + x_2 + x_3 = 0. \tag{5}$$

The condition that (3) reduce to (2) is $a_{11} = a_{22} = a_{33}$. This is also the condition that (4) reduce to (5). Accordingly, we meet the original demand by requiring that the unit line d be so chosen that its equation (4) takes on the form (5).

A means of satisfying this requirement is obvious. We have merely to take as d the line of the three collinear points $(0, 1, -1)$, $(-1, 0, 1)$, $(1, -1, 0)$. For, since the coordinates of these points must satisfy (4), we have immediately $a_{11} = a_{22} = a_{33}$; consequently, (4) reduces to (5) and the condition (3) takes on the desired form (2).

Geometrical Relationship between Unit Point and Unit Line. The points $(0, 1, -1)$, $(-1, 0, 1)$, $(1, -1, 0)$ are the points D_1, D_2, D_3 in which d meets the sides of the triangle of reference. Since they are collinear and lie respectively on the sides of the triangle, the lines joining their harmonic conjugates, with respect to the vertices of the triangle, to the opposite vertices are concurrent. These lines are readily shown to be the lines d_1, d_2, d_3, concurrent in the unit point D.

Thus, if the unit point D is given and the lines d_1, d_2, d_3 are drawn meeting the opposite sides of the triangle in C_1, C_2, C_3, the points D_1, D_2, D_3 which are the harmonic conjugates of C_1, C_2, C_3 with respect to the vertices are collinear, and the line of collinearity is the unit line d.

Relationship of Point and Line in Projective Coordinates. In employing projective point and line coordinates simultaneously, we shall always assume that the systems of reference have been chosen as described, so that the condition that the point x lie on the line u is

$$u_1x_1 + u_2x_2 + u_3x_3 = 0. \tag{2}$$

The theory of the relationship of point and line (Ch. V, § 4) has, then, the same analytic form in terms of projective coordinates as in terms of metric coordinates. From (2) it follows that $(a\,|\,u) = 0$ is the equation of the point a and that $(a\,|\,x) = 0$ is the equation of the line a, and hence the entire theory in question remains unchanged.

EXERCISE. The triangles $A_1A_2A_3$ and $C_1C_2C_3$ (Fig. 4) are in the relationship of Desargues, inasmuch as the lines joining corresponding vertices are concurrent in the unit point D. Prove that the line of collinearity of the points of intersection of corresponding sides is the unit line d.

5. Two-Dimensional Projective Transformations. The totality of points in the plane and the totality of lines in the plane we shall call *two-dimensional fundamental forms.*

DEFINITION. *A transformation of the elements of a two-dimensional fundamental form F into the elements of a two-dimensional fundamental form F' is projective if it* (*a*) *establishes a one-to-one correspondence between the elements of F and F',* (*b*) *establishes a one-to-one correspondence between the one-dimensional forms composed of elements of F and the one-dimensional forms composed of elements of F', and* (*c*) *preserves cross ratio.*

It will be well to paraphrase the definition in a particular case. A

transformation of the points of a plane of points into the lines of a plane of lines is projective if it establishes (*a*) a one-to-one correspondence between the points of the first plane and the lines of the second, and (*b*) a one-to-one correspondence between the ranges of points in the first plane and the pencils of lines in the second, and (*c*) preserves cross ratio.

THEOREM 1. *There exists a unique projective transformation of one two-dimensional form into a second which orders to four given elements of the first form four prescribed elements of the second, provided no three of either set of four elements are linearly dependent.*

Thus, a projective transformation of a plane of points into a plane of lines is uniquely determined when to four points of the first plane, no three of which are collinear, are ordered four lines of the second plane, no three of which are concurrent.

We prove the theorem in this case. Take the four points of the first plane as the basic points A_1, A_2, A_3, D of a system of projective point coordinates (x_1, x_2, x_3), and the corresponding four lines of the second plane as the basic lines a'_1, a'_2, a'_3, d' of a system of projective line coordinates. It can then be shown (Ex. 2) that, if there exists a projective transformation which orders A_1 to a'_1, A_2 to a'_2, A_3 to a'_3, D to d', the transformation is unique and is represented by the equations

$$(1) \qquad \rho u'_1 = x_1, \qquad \rho u'_2 = x_2, \qquad \rho u'_3 = x_3.$$

That this transformation is projective is evident from the fact that the equations are those of a linear transformation of the x's into the u''s. Hence the theorem is proved.

The projective transformation (1) of the plane of points into the plane of lines is linear and will remain linear regardless of the systems of projective coordinates employed in the two planes. Conversely, every linear transformation of projective point coordinates (x_1, x_2, x_3) in the first plane into projective line coordinates (u'_1, u'_2, u'_3) in the second plane we know has the properties (*a*), (*b*), (*c*) of our definition and is therefore a projective transformation.

THEOREM 2. *The projective transformations of one two-dimensional form into a second are identical with the linear transformations of projective coordinates for the elements of the first form into projective coordinates for the elements of the second.*

EXERCISES

1. Give *in detail* the proof of Theorem 1 in the case of two planes of points.
2. The same, for a plane of points and a plane of lines.

6. Collineations and Correlations. There are four kinds of projective transformations of a plane M into a plane M':

A_1. Those of the points of M into the points of M'.

A_2. Those of the lines of M into the lines of M'.

B_1. Those of the points of M into the lines of M'.

B_2. Those of the lines of M into the points of M'.

Collineations. In Ch. VII, § 13 we proved that the transformations A_1 and A_2 are in effect identical and we agreed to give them the common name "collineations."

Correlations. The transformations B_1 and B_2 are also identical in that the relationships which they establish between the two planes are the same; B_1 are the transformations of the points of M into the lines of M' which (*a*) establish a one-to-one correspondence between the points of M and the lines of M', (*b*) establish a one-to-one correspondence between the lines of M and the points of M', and (*c*) preserve cross ratio; and B_2 are the transformations of the lines of M into the points of M' which have the same three properties with the order of (*a*) and (*b*) reversed.

The transformations B_1 and B_2 are known as *correlations.* Thus, *a correlation is a projective transformation of a plane M into a plane M′ which carries the points of M into the lines of M′ and hence the lines of M into the points of M′, or vice versa.*

A correlation carries a point into a line and a line into a point; collinear points into concurrent lines, and vice versa; a cross ratio of four collinear points into an equal cross ratio of four concurrent lines, and vice versa. In short, a correlation always carries a given figure into the dual figure, a given property into the dual property. It is the analytic representation of the process of dualization. It *correlates* the configurations and theorems of point geometry with those of line geometry.

A correlation can be represented analytically in two ways, either as a transformation B_1 or as a transformation B_2. The general correlation and its inverse, expressed in terms of point coordinates in M and line coordinates in M', are

$$(1) \quad \rho u_i' = \sum_{j=1}^{3} a_{ij}x_j, \qquad \sigma x_i = \sum_{j=1}^{3} A_{ji}u_j', \quad (i = 1, 2, 3), \quad |a_{ij}| \neq 0,$$

where A_{ij} is the cofactor of a_{ij} in $|a_{ij}|$. The same correlation and its inverse, written in terms of line coordinates in M and point coordinates in M', are

$$(2)\qquad \lambda x'_i = \sum_{j=1}^{3} A_{ij}u_j, \qquad \mu u_i = \sum_{j=1}^{3} a_{ji}x'_j, \qquad (i = 1, 2, 3).$$

If M and M' are the same plane and x, u and x', u' are referred to the same system of reference in this plane, equations (1) represent the general correlation of the plane.

The correlations of the plane do not, in themselves, form a group; for the product of two of them is evidently a collineation. The collineations and correlations of the plane, taken together, do form a group, the *general* projective group of the plane.

EXERCISES

1. Deduce equations (2) from equations (1).

2. Write the equations of the general correlation expressed as a transformation B_2 and deduce from them the equations of the correlation as a transformation B_1.

3. Show that the correlation

$$\rho u'_1 = 2x_1 - x_2 + x_3, \qquad \rho u'_2 = -x_1 + x_2 + 3x_3, \qquad \rho u'_3 = x_1 + 3x_2 - x_3$$

is involutory, that is, that carrying it out twice results in the identical transformation.

4. Prove that the locus of a point which moves so that it always lies on the line corresponding to it by the correlation of Ex. 3 is a nondegenerate conic.

5. The determinant of the correlation of Ex. 3 is symmetric; see Ch. I, § 6, Ex. 3. Show that the properties established for it in Exs. 3, 4 are possessed by every correlation whose determinant is symmetric.

7. Trilinear Coordinates. Let us consider equations (2) of § 2, namely

$$(1)\qquad \rho x'_1 = \frac{|x\,b\,c|}{|d\,b\,c|}, \qquad \rho x'_2 = \frac{|x\,c\,a|}{|d\,c\,a|}, \qquad \rho x'_3 = \frac{|x\,a\,b|}{|d\,a\,b|},$$

with the idea of finding a *metric* interpretation of the projective coordinates (x'_1, x'_2, x'_3) which are defined by them. In so doing, we assume that the vertices a, b, c of the triangle of reference and the unit point d are finite points and restrict ourselves in the beginning to the case of a finite point P.

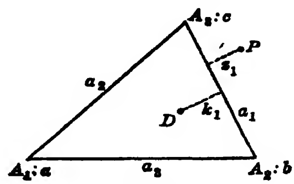


FIG. 5

The determinants $|x\,b\,c|$ and $|d\,b\,c|$ are essentially the numerators in the formulas for directed distances from the line a_1 to the points P and D. In fact, if we assume that $x_3 = 1$ and $d_3 = 1$, these directed distances are

$$(2) \qquad z_1 = \pm\frac{1}{A}\,|x\,b\,c|, \qquad k_1 = \pm\frac{1}{A}\,|d\,b\,c|,$$

where

$$A^2 = (b_2c_3 - b_3c_2)^2 + (b_3c_1 - b_1c_3)^2 + (b_1c_2 - b_2c_1)^2, \quad A > 0,$$

and their ratio is the ratio of the two determinants.

In the same way we can rewrite the other two ratios in (1). Hence we can rewrite (1) in the form

$$(3) \qquad \rho x_1' = \frac{1}{k_1}z_1, \qquad \rho x_2' = \frac{1}{k_2}z_2, \qquad \rho x_3' = \frac{1}{k_3}z_3,$$

where k_i is a *chosen* directed distance from a_i to D and z_i is the *corresponding* directed distance from a_i to P.

THEOREM 1. *The projective coordinates of a finite point P, interpreted metrically, are proportional to multiples of directed distances from the sides of the triangle of reference to P; the multipliers are the reciprocals of the corresponding directed distances from the sides to the unit point.*

Considered from this point of view, the coordinates (x_1', x_2', x_3') are known as *trilinear* coordinates of P.

Barycentric Coordinates. The first system of trilinear coordinates was developed by Moebius by means of an ingenious mechanical idea. He conceived, as the coordinates of a point P, the three masses m_1, m_2, m_3 which must be placed at the vertices of the triangle in order that P become the center of gravity of the triangle. To show that these are actually trilinear coordinates, we place at the point P the mass $m = m_1 + m_2 + m_3$. Since P is the center of gravity, the sum of the algebraic moments of the masses m_1, m_2, m_3 about a side of the triangle is equal to the moment of the mass m about this side. Hence (Fig. 6):

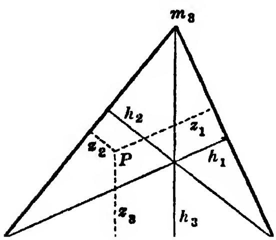


FIG. 6

$$h_1m_1 = mz_1, \qquad h_2m_2 = mz_2, \qquad h_3m_3 = mz_3,$$

or

$$m_1 : m_2 : m_3 = \frac{1}{h_1} z_1 : \frac{1}{h_2} z_2 : \frac{1}{h_3} z_3.$$

But the altitudes h_1, h_2, h_3 of the triangle are proportional to the distances from the sides to the intersection of the medians. Hence, the barycentric coordinates are the trilinear coordinates obtained by taking the intersection of the medians as the unit point.*

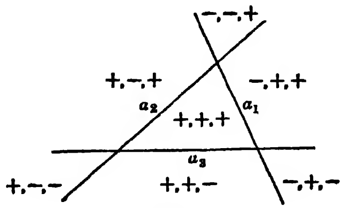


FIG. 7

Center of Inscribed Circle as Unit Point. If we choose the directed distances from the sides of the triangle so that k_1, k_2, k_3 are all positive,† and take as D the center of the inscribed circle, k_1, k_2, k_3 are all equal and the directed distances z_1, z_2, z_3 of a point P from the sides of the triangle are themselves coordinates of P. Their signs in the various regions of the plane are shown in Fig. 7.

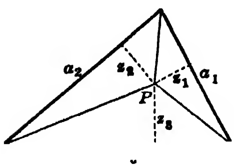


FIG. 8

If a_1, a_2, a_3 now denote the lengths of the sides of the triangle and A its area, then

(4) $$a_1 z_1 + a_2 z_2 + a_3 z_3 = 2A.$$

This relation is evident from Fig. 8 if P is inside the triangle and, by use of the facts supplied by Fig. 7, it is readily seen to hold when P is outside the triangle.

The points for which the expression on the left-hand side of (4) vanishes, since they cannot be finite, must be ideal. Hence the equation of the line at infinity is

$$a_1 z_1 + a_2 z_2 + a_3 z_3 = 0$$

or

$$z_1 \sin A_1 + z_2 \sin A_2 + z_3 \sin A_3 = 0.$$

In discussing metric properties of a triangle in Ch. III, § 10, we took the origin of coordinates inside the triangle and introduced the equations in normal form, $\alpha = 0$, $\beta = 0$, $\gamma = 0$, of the sides of the

* The proof apparently applies only to the case when m_1, m_2, m_3 are all positive. Actually, if it is properly interpreted, it is valid in all cases except when $m = 0$. What are the facts when $m = 0$?

† For example, we choose the upper or lower sign in (2) according as to which makes $k_1 > 0$.

triangle. The values of α, β, γ for a point P are then the directed distances from the sides of the triangle to P and are all of the same sign when P is inside the triangle. Hence α, β, γ are trilinear coordinates of P of the specialized type we are now considering. Accordingly, the results obtained in terms of them will all be valid here when we replace α, β, γ by z_1, z_2, z_3.

For example, it follows from Th. 4 of the paragraph cited that the equations of the medians of the triangle are

$$z_1 \sin A_1 = z_2 \sin A_2, \quad z_2 \sin A_2 = z_3 \sin A_3, \quad z_3 \sin A_3 = z_1 \sin A_1.$$

Hence coordinates of the point of intersection of the medians are

$$(\sin A_2 \sin A_3, \sin A_3 \sin A_1, \sin A_1 \sin A_2).$$

EXERCISES

Exercises 1–6 bear on the specialized coordinates last developed.

1. Show that the centers of the inscribed and escribed circles are $(1, 1, 1)$, $(-1, 1, 1)$, $(1, -1, 1)$, $(1, 1, -1)$.

2. Prove that the point of intersection of the altitudes has the coordinates $(\cos A_2 \cos A_3, \cos A_3 \cos A_1, \cos A_1 \cos A_2)$.

3. Show that $(\cos A_1, \cos A_2, \cos A_3)$ is the center of the circumscribed circle.

4. Prove that the intersection of the medians, the intersection of the altitudes, and the center of the circumscribed circle are collinear.

5. Show that the coordinates of the point P_2 which is isogonally conjugate (Ch. III, § 10, Ex. 5) to the point $P_1 : (m_1, m_2, m_3)$ are $(1/m_1, 1/m_2, 1/m_3)$. It is assumed that P_1 is not on a side of the triangle.

6. Prove that the equation of the circumscribed circle is

$$z_2 z_3 \sin A_1 + z_3 z_1 \sin A_2 + z_1 z_2 \sin A_3 = 0.$$

7. Develop a metric interpretation of projective line coordinates.

EXERCISES ON CHAPTER X

1. A transformation of the points of a plane leaves fixed a certain point A and every point on a certain line a not passing through A and carries any other point P into its harmonic conjugate with respect to A and the point in which AP meets a. Find the equations of the transformation in terms of suitably chosen projective coordinates and hence show that the transformation is projective and involutory.

2. Characterize geometrically the linear transformation

$$\rho x_1' = x_1, \qquad \rho x_2' = x_2, \qquad \rho x_3' = k x_3, \qquad k \neq 0, 1,$$

as a transformation of the plane into itself.

3. Show that, if a collineation possesses four fixed points, no three of which are collinear, it is the identity.

4. A collineation has three real noncollinear points as its only fixed points. Show that the equations of the collineation, referred to suitably chosen projective coordinates, are

$$\rho x_1' = a_1 x_1, \qquad \rho x_2' = a_2 x_2, \qquad \rho x_3' = a_3 x_3,$$

where no one of the constants a_1, a_2, a_3 is zero and no two are equal. Find the corresponding equations in line coordinates and show that the collineation leaves fixed just three lines.

5. A collineation, not the identity, leaves fixed every point on a certain line and every line through a certain point on this line. Show that projective point coordinates can be so chosen that it is represented by the equations

$$\rho x_1' = x_1 + c x_2, \qquad \rho x_2' = x_2, \qquad \rho x_3' = x_3, \qquad c \neq 0.$$

6. Establish the theorem: *The point r is a fixed point of the collineation*

$$\rho x_i' = \sum_{j=1}^{3} a_{ij} x_j, \qquad (i = 1, 2, 3), \quad |a_{ij}| \neq 0,$$

if and only if (r_1, r_2, r_3) is a solution of the system

$$\begin{aligned} (a_{11} - \rho)x_1 + a_{12}x_2 + a_{13}x_3 &= 0, \\ a_{21}x_1 + (a_{22} - \rho)x_2 + a_{23}x_3 &= 0, \\ a_{31}x_1 + a_{32}x_2 + (a_{33} - \rho)x_3 &= 0, \end{aligned}$$

where ρ is a root of the cubic equation

$$\begin{vmatrix} a_{11} - \rho & a_{12} & a_{13} \\ a_{21} & a_{22} - \rho & a_{23} \\ a_{31} & a_{32} & a_{33} - \rho \end{vmatrix} = 0.$$

7. Find the fixed points and the fixed lines of each of the collineations:

$$\begin{aligned} \rho x_1' &= \phantom{-3x_1 +{}} - 2x_2 + 2x_3, \\ \rho x_2' &= -3x_1 + x_2 + 3x_3, \\ \rho x_3' &= -x_1 + x_2 + 3x_3; \end{aligned} \qquad \begin{aligned} \rho x_1' &= 3x_1 + x_2 - x_3, \\ \rho x_2' &= x_1 + x_2 + x_3, \\ \rho x_3' &= 2x_1 \phantom{{}+x_2} + 2x_3. \end{aligned}$$

CHAPTER XI

GEOMETRIES

1. Counting Constants. If the number of objects in a set is finite, it is a simple matter to count them. It is an entirely different problem to measure the number of objects in a set if that number is infinite. At times it suffices to say simply that the number is infinite. But this is not always a sufficient statement of the facts. There are, for example, infinitely many points on a line and infinitely many points in a plane, yet we should hardly think of the one set as comparable with the other. We need, then, a classification of infinities.

The infinities with which we have to deal in geometry are in general characterized by the fact that they admit analytic representation by means of coordinates, or parameters (arbitrary constants).

For example, the points in a plane can be represented by means of two (Cartesian) coordinates x, y. Thus, values assigned to two parameters are sufficient to determine a point in the plane. Moreover, two is the *smallest* number of parameters which suffice for this purpose.

A circle in the plane can be determined by values assigned to three parameters, say, the coordinates (x_0, y_0) of the center and the radius r. Furthermore, it is impossible to fix the position and the size of a circle by values assigned to less than three parameters.

A point on the parabola $y = x^2$ has two coordinates x, y. The two coordinates are, however, *not independent;* when x is given, y is determined. Thus the single parameter x suffices to determine a point on the parabola.

As a further illustration, let us count the parameters sufficient to determine an arbitrary triple of collinear points in the plane. Two parameters suffice to fix each of the three points. The six parameters thus involved are not independent; they are connected by the relation which expresses the fact that the points are collinear. Hence, five is the smallest number of parameters sufficient to determine a collinear point-triple in the plane.

DEFINITION. *If the smallest number of parameters sufficient to determine the general element of a set of infinitely many elements is n, the set*

is said to contain ∞^n *(infinity n) elements or to form an n-parameter family of elements.**

According to the definition, there are in the plane: ∞^2 points or a two-parameter family of points; ∞^3 circles or a three-parameter family of circles; and ∞^6 collinear point-triples.

The parameters of the smallest set of parameters sufficient to determine the general element of an infinite set are all essential for the determination of the element. Accordingly, we shall frequently refer to them, for the sake of brevity, as *essential parameters.*

Homogeneous Parameters. A given point of the plane has unique nonhomogeneous coordinates (x, y), but infinitely many sets of homogeneous coordinates (x_1, x_2, x_3) of the form $(\rho x_1, \rho x_2, \rho x_3)$, where ρ is an arbitrary constant, $\neq 0$. Thus, to the point corresponds just one pair of nonhomogeneous coordinates, but a one-parameter family of sets of homogeneous coordinates, where the parameter is the factor of proportionality ρ.

Again, an arbitrary circle in the plane can be thought of as determined by the three nonhomogeneous parameters A_1, A_2, A_3 which appear as coefficients in the equation

$$x^2 + y^2 + A_1x + A_2y + A_3 = 0,$$

or by the four homogeneous parameters a_0, a_1, a_2, a_3 obtained by rewriting this equation in the form

$$a_0(x^2 + y^2) + a_1x + a_2y + a_3 = 0, \qquad a_0 \neq 0.$$

The latter parameters are homogeneous, inasmuch as the circle remains the same when a_0, a_1, a_2, a_3 are replaced by ρa_0, ρa_1, ρa_2, ρa_3. It is evident that to a given circle there corresponds ∞^1 sets of values of the homogeneous parameters, but only one set of values of the nonhomogeneous parameters.

In counting the homogeneous parameters in each of these cases, we count each element (point or circle) an infinity of times equivalent to one parameter. Consequently, we can obtain the essential number of parameters in each case by subtracting one from the number of

* We are, of course, restricting ourselves to parametric representations of the elements of the set which are continuous, that is, which establish a *continuous* correspondence between the elements of the set and the sets of values of the parameters. Every representation by means of coordinates or parameters with which the student is familiar, and, in fact, every representation which is of importance in geometry, has this property.

homogeneous parameters. Thus, there are in the case of the point $3 - 1 = 2$, and in the case of the circle $4 - 1 = 3$, essential parameters.

Mixed Parameters. The set of parameters x_1, x_2, x_3, r for a circle, where x_1, x_2, x_3 are the homogeneous coordinates of the center and r is the radius, is neither homogeneous nor nonhomogeneous. It consists of the three homogeneous parameters x_1, x_2, x_3 and the nonhomogeneous parameter r, and may fittingly be described as a *mixed* set of parameters.

Though the parameters are not homogeneous, there corresponds to each element, as in the case of homogeneous parameters, ∞^1 sets of parametric values: ρx_1, ρx_2, ρx_3, r. Hence the number of essential parameters is $4 - 1 = 3$; there are ∞^3 circles in the plane.

A point-triple in the plane can be determined by the homogeneous coordinates a_1, a_2, a_3; b_1, b_2, b_3; c_1, c_2, c_3 of the points. Here we have, not one set of nine homogeneous parameters, but three sets of three homogeneous parameters each, which are independent in that the three corresponding factors of proportionality can be chosen independently of one another. Hence, to a given point-triple correspond ∞^3 sets of parametric values: ρa_1, ρa_2, ρa_3; σb_1, σb_2, σb_3; τc_1, τc_2, τc_3. The number of essential parameters is therefore $9 - 3 = 6$; there are ∞^6 point triples in the plane.

Miscellaneous Examples. Let it be required to find the number of lines in the plane when an arbitrary line is thought of as determined by two points.

Four parameters, two for each point, suffice to fix the line. Thereby we have counted the line as often as there are pairs of points on it, namely, ∞^2 times. The number of essential parameters is then $4 - 2 = 2$; there are ∞^2 lines in the plane.

Let it be required to count the number of lines in space, a line in space to be thought of as determined by its direction components and a point on it.

A point in space has three nonhomogeneous coordinates, and a line in space, three direction components. But the direction components are homogeneous and the point can be taken in any one of the ∞^1 positions on the line. Hence, there are ∞^4 lines in space.

Real and Complex Parameters. The reader has doubtless assumed that we have been dealing with real elements and counting real

parameters. He might equally well have thought of both the elements and the parameters as complex. The results would remain the same. For example, the general real point in the plane depends on two essential real parameters, and the general complex point, on two essential complex parameters.

The situation is different, however, if we are considering *complex* elements and count *real* parameters. There are, then, ∞^4 complex points in the plane, for the coordinates $(x' + i\,x'',\ y' + i\,y'')$ of a complex point involve four real parameters.

With the count of real parameters as a basis we are able to compare the number of complex points in the plane with the number of real points. Whereas there are but ∞^2 real points, there are ∞^4 complex points. In other words, the complex points outnumber the real points to the same degree that the real lines in space outnumber the real lines in a plane!

EXERCISES

In each of the following examples, determine the number of the elements described. Unless the contrary is stated, the elements and parameters are to be assumed as both real (or both complex).

1. Pairs of points in the plane.

2. Triangles in the plane, when a triangle is thought of as determined by three nonconcurrent lines.

3. Complete quadrangles in the plane, if a complete quadrangle is considered as determined by (*a*) four points, no three collinear; (*b*) six lines passing by threes through four points.

4. Circles in a plane, if a circle is considered as determined by three noncollinear points.

5. Conics in a plane.

6. Ellipses in a plane, if an ellipse is considered as determined by its foci and eccentricity.

7. Parabolas in the plane, when a parabola is given by (*a*) its equation; (*b*) its definition in terms of focus and directrix.

8. Points in space.

9. Planes in space, when a plane is determined by (*a*) its equation; (*b*) three noncollinear points; (*c*) a line and a point.

10. Lines in space, when a line is determined by (*a*) two points; (*b*) two planes.

11. Points of a sphere.

12. Spheres in space.

13. Circles in space.

14. Conics in space.

15. Lines through a point.

16. Planes through a point.

17. Complete quadrangles with a given diagonal triangle, in the plane.

18. Complex points on a line, when real parameters are counted.

2. Dimensionality. In saying that a line is one-dimensional, a plane two-dimensional, and space three-dimensional, we mean simply that a line contains ∞^1 points, a plane ∞^2 points, and space ∞^3 points. Thereby, we are assuming the classical point of view that the point is the fundamental element and that line, plane, and space are to be thought of as consisting of points.

If the line is taken as the fundamental element, we must count the number of lines in a given configuration or manifold, to determine its dimensionality. The plane, considered as the totality of its lines, is still two-dimensional. But space, since it contains ∞^4 lines, is four-dimensional. The line itself fails of dimensionality; the linear one-dimensional manifold generated by lines is the pencil of lines, or the point.

EXERCISES

1. If the plane is chosen as the fundamental element, what is the dimensionality of space? Of a plane? Of a line? Of a point?

2. If the circle is taken as the fundamental element, what is the dimensionality of the plane? Of space?

3. Geometries. In building a geometry, we begin by choosing a fundamental element. We have already recognized that the line can be used as the fundamental element as well as the point. Thus we have line geometry as well as point geometry. Similarly, there is circle geometry, in which the circle is the fundamental element.

The dimensionality of a geometry is the dimensionality of the totality, or manifold, of fundamental elements studied. For example, point geometry and line geometry in the plane are both two-dimensional. Point geometry of a line is one-dimensional; so also is line geometry of a point.

Point geometry and plane geometry in space are three-dimensional, since there are in space ∞^3 points and ∞^3 planes. Line geometry in space is four-dimensional; it furnishes us with an important example of a geometry which is in dimension greater than three and yet is within the limits of our powers of visualization.

The choice of the fundamental element and of the manifold of these elements to be discussed constitutes but part of the construction of a geometry. There remains the important question as to the nature of

the geometric properties to be studied. This question is settled by the choice of a group of transformations on the coordinates of the fundamental element of the manifold. The geometry is then to consist of the discussion of those properties which are invariant with respect to the group.

For example, if the points of the plane are to be discussed, we could take as the group of transformations the totality of all transformations of the points of the plane

$$x' = f(x, y), \qquad y' = \phi(x, y)$$

which are one-to-one and continuous. The resulting geometry is known as *analysis situs.*

In the present book we deal primarily with the point geometry and the line geometry of the plane in which the governing group of transformations is the projective group or a subgroup of the projective group.

4. Affine Geometry. The plane of metric geometry contains one exceptional line and, on this line, two exceptional points, and the transformations which control metric geometry, namely the rigid motions, all leave the line and each of the two points invariant. On the other hand, the projective plane has no exceptional point or line and, correspondingly, there is no point or line which is left fixed by all collineations.

Intermediate between these geometries is the geometry which has merely an exceptional line and is governed by the particular collineations which transform this line into itself. These collineations are known as *affine transformations*, and the geometry, as *affine geometry.*

Affine geometry may be considered as a generalization of metric geometry or as a "subgeometry" of projective geometry. In developing it, we shall employ, on the one hand, the terminology of metric geometry and, on the other hand, special projective coordinates so chosen that the fixed line, the line at infinity, is always the line $x_3 = 0$. The affine transformations are, then, the collineations of the plane which carry the line at infinity: $x_3 = 0$ into itself, or, what is the same thing, carry a finite point always into a finite point.

The collineation,

$$\rho x_i' = \sum_{j=1}^{3} a_{ij}x_j, \qquad (i = 1, 2, 3), \qquad |a_{ij}| \neq 0,$$

is an affine transformation if and only if $a_{31} = a_{32} = 0$; for, it carries

every ideal point into an ideal point if and only if x_3' vanishes whenever x_3 is zero, regardless of the values of x_1 and x_2, that is, if and only if $a_{31}x_1 + a_{32}x_2 \equiv 0$.

Thus, the general affine transformation is

$$\begin{aligned} \rho x_1' &= a_{11}x_1 + a_{12}x_2 + a_{13}x_3, \\ \rho x_2' &= a_{21}x_1 + a_{22}x_2 + a_{23}x_3, \\ \rho x_3' &= \phantom{a_{11}x_1 + a_{12}x_2 +} a_{33}x_3, \end{aligned} \qquad a_{33}\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \neq 0,$$

or in terms of the nonhomogeneous coordinates $x = x_1/x_3$, $y = x_2/x_3$,

$$\begin{aligned} x' &= a_1x + b_1y + c_1, \\ y' &= a_2x + b_2y + c_2, \end{aligned} \qquad \Delta = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \neq 0, \tag{1}$$

where the six coefficients are subject merely to the restriction that $\Delta \neq 0$.

THEOREM 1. *The totality of affine transformations forms a group, a subgroup of the group of collineations.*

An invariant peculiar to the group of affine transformations is obtained by specializing the fundamental projective invariant, cross ratio. Three distinct points P_1, P_2, P on a line L and the ideal point P_∞ on L are carried by the general affine transformation into three distinct points P_1', P_2', P' on a line L' and the ideal point P_∞' on L'. Hence

$$(P_1'P_2',\ P'P_\infty') = (P_1P_2,\ PP_\infty).$$

Since $\overline{P_\infty'P_1'}/\overline{P_\infty'P_2'} = 1$ and $\overline{P_\infty P_1}/\overline{P_\infty P_2} = 1$, the equation becomes

$$\frac{\overline{P'P_1'}}{\overline{P'P_2'}} = \frac{\overline{PP_1}}{\overline{PP_2}}.$$

THEOREM 2. *The ratio in which the point P divides the line-segment P_1P_2 is an invariant with respect to the group of affine transformations.*

Inasmuch as this ratio is not invariant with respect to all collineations, we call it an *affine invariant*, and the corresponding geometric property, an affine property.

THEOREM 3. *Parallelism of lines is an affine property.*

Affine coordinates are the special projective coordinates obtained by taking always, as the side $x_3 = 0$ of the triangle of reference, the line at infinity. Inasmuch as affine geometry deals primarily with the finite points of the plane, affine coordinates are in general

most useful in the nonhomogeneous form

$$(2) \qquad x = \frac{x_1}{x_3}, \qquad y = \frac{x_2}{x_3}.$$

The transformations from one system of affine coordinates to a second are the transformations of projective coordinates which do not change the side $x_3 = 0$ of the triangle of reference. Consequently, they are the linear transformations which carry $x_3 = 0$ into itself, namely, the transformations (1). The transformations (1) may, then, be regarded, not only as the affine transformations of the plane, but also as the transformations from one pair of affine coordinates (x, y) to a second pair (x', y').

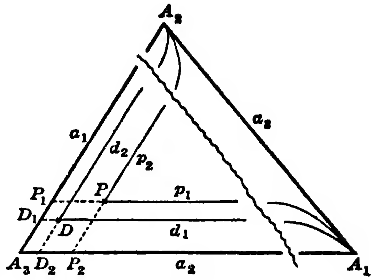


FIG. 1

It is of interest to interpret the affine coordinates (2) from the point of view of metric geometry. By definition, Ch. X, § 1,

$$x = \frac{x_1}{x_3} = (a_3a_1,\ d_2p_2), \qquad y = \frac{x_2}{x_3} = (a_3a_2,\ d_1p_1).$$

Hence (Fig. 1),

$$x = (A_1A_3,\ D_2P_2), \qquad y = (A_2A_3,\ D_1P_1),$$

or

$$x = (P_2D_2,\ A_3A_1), \qquad y = (P_1D_1,\ A_3A_2).$$

Finally, since A_1 and A_2 are points at infinity,

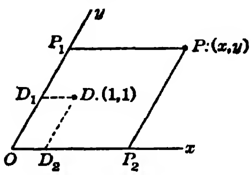


FIG. 2

$$x = \frac{\overline{A_3P_2}}{\overline{A_3D_2}}, \qquad y = \frac{\overline{A_3P_1}}{\overline{A_3D_1}},$$

or (Fig. 2),

$$(3) \qquad x = \frac{\overline{OP_2}}{\overline{OD_2}}, \qquad y = \frac{\overline{OP_1}}{\overline{OD_1}}.$$

Thus, x and y are metric coordinates of the point P referred to oblique axes, or *oblique Cartesian coordinates.*

This interpretation of affine coordinates is instructive in that it lends visible emphasis to the fact that angle has no place in affine geometry. It may, however, be misleading in that it appears, on first sight, to assign to distance a leading rôle. Distance itself is not

an affine property. In fact, only the ratios of distances which lie on the same line or on parallel lines have significance in affine geometry (see Th. 2 and Ex. 3), and it is merely in the form of ratios of this kind that distance appears in equations (3).

EXERCISES

1. Prove Theorem 1.

2. Establish Theorem 3.

3. Prove that the ratio of two directed distances which are on the same line or on parallel lines is an affine invariant; show that these are the only cases in which a ratio of distances is unchanged by all affine transformations.

4. Show that the area of a triangle is a relative invariant with respect to the group of affine transformations.

5. The equations

$$x' = 3x - 2y - 1, \qquad y' = x + 3y - 4$$

represent a change of affine coordinates. Find the original coordinates of the new basic points and the equations in x and y of the new axes.

6. Determine the most general transformation of affine coordinates which introduces the lines

$$2x - y = 0, \qquad x + y + 1 = 0$$

as the new axes $x' = 0$, $y' = 0$, and hence find the particular transformation which also introduces $(3, 1)$ as the new unit point.

7. A projective transformation of a line L into a second line L' is called an affine transformation if it carries the point at infinity on L into the point at infinity on L'. Show that the general affine transformation of L into L' is identical with the general transformation of similarity of L into L', namely $x' = ax + b$, $a \neq 0$.

5. Metric Aspects of Affine Transformations.

THEOREM 1. *An affine transformation multiplies by the same constant factor, k, the lengths of all line-segments with the same direction.*

Since the correspondence established by the given affine transformation between the points of two corresponding lines, L and L', is affine and hence identical with a transformation of similarity (§ 4, Ex. 7), the affine transformation multiplies the length of every line-segment on L by the same constant factor k. If AB is an arbitrary segment on L, and $A'B'$ the corresponding segment on L', $A'B' = k \cdot AB$.

Let M and M' be any pair of corresponding lines parallel respectively to L and L', and let CD and $C'D'$ be corresponding segments on M and M'. If CD is chosen equal to AB, so that $ABDC$ is a parallelogram, then $A'B'D'C'$ must also be a parallelogram, and $C'D' = A'B'$. Hence, $C'D' = k \cdot CD$. Thus the constant factor k applies to distances on M as well as to those on L, and the theorem is proved.

It follows that we can find k for all line-segments by finding it for the radii of a single circle C. Since an affine transformation carries a conic into a conic and finite points into finite points, it must carry the circle C into an ellipse E. Moreover, it carries the center of C into the center of E (§ 4, Th. 2). Hence it carries a radius r of C into a semi-diameter d of E. Thus, if C is a circle of unit radius, the factor k by which all distances in the direction of the radius r are multiplied is precisely the length of the semi-diameter d.

As r takes on the positions of all the radii of C, d takes on the positions of all the semi-diameters of E. Hence k varies from the smallest semi-diameter of E to the largest, that is, from the semi-minor axis b to the semi-major axis a:

$$b \leqq k \leqq a.$$

It is evident that k takes on the value unity if and only if unity lies in the closed interval from b to a.* Hence:

THEOREM 2. *There are two, one, or no pencils of real parallel lines which are transformed by an affine transformation so that distance is preserved, according as unity lies within, is an end-point of, or lies without the interval from b to a, where b and a are the semi-axes of an ellipse which corresponds by the transformation to a circle of unit radius.*

Incidentally we have also established the following facts.

THEOREM 3. *The ellipses into which the circles of the plane are carried by an affine transformation are all similar and similarly placed. Their transverse and conjugate axes have the directions corresponding to those for which k has respectively its maximum and minimum values.*

The circle C of unit radius with center at O, namely

$$x = \cos\theta, \qquad y = \sin\theta,$$

is carried by the affine transformation

$$(1) \qquad x' = a_1x + b_1y + c_1, \qquad y' = a_2x + b_2y + c_2$$

into the ellipse E,

$$x' - c_1 = a_1\cos\theta + b_1\sin\theta, \qquad y' - c_2 = a_2\cos\theta + b_2\sin\theta,$$

with center at (c_1, c_2). Hence the value of k^2 for the direction of slope-angle θ is

$$k^2 = (a_1^2 + a_2^2)\cos^2\theta + 2(a_1b_1 + a_2b_2)\cos\theta\sin\theta + (b_1^2 + b_2^2)\sin^2\theta.$$

* The reader should draw figures of C and E illustrating the various cases listed in the theorem.

By means of this formula the foregoing theory may be applied in any given case.

We have restricted ourselves to affine transformations of general type, excluding the case in which k^2 is a constant. In this case distance is a relative invariant and the affine transformation is, in particular, a transformation of similarity, combined perhaps with a reflection in the axis of x. It is readily found that $dk^2/d\theta \equiv 0$ if and only if

$$(2) \qquad a_1^2 + a_2^2 = b_1^2 + b_2^2, \qquad a_1b_1 + a_2b_2 = 0.$$

But these conditions are equivalent to the relations

$$(3) \qquad b_1 = \mp a_2, \qquad b_2 = \pm a_1,$$

which characterize (1) as a transformation of similarity (Ch. VIII, § 7), or the product of a transformation of similarity and a reflection in the x-axis.

EXERCISES

1. Determine for each of the following affine transformations the directions for which k has its extrema, the directions of the axes and the eccentricity of the ellipses into which circles are carried, and when they exist, the directions for which distance is preserved.

(a) $\qquad x' = -x - 3y, \qquad y' = 3x + y;$

(b) $\qquad x' = x + \frac{1}{2}y + 1, \qquad y' = x - \frac{1}{2}y - 2.$

2. Show that equations (3) are equivalent to equations (2).

3. Prove that the only affine transformations which preserve directed angle are the transformations of similarity.

6. Projective, Affine, and Metric Geometries. The group of collineations,

$$G_8 \qquad \begin{aligned} \rho x_1' &= a_{11}x_1 + a_{12}x_2 + a_{13}x_3, \\ \rho x_2' &= a_{21}x_1 + a_{22}x_2 + a_{23}x_3, \\ \rho x_3' &= a_{31}x_1 + a_{32}x_2 + a_{33}x_3, \end{aligned} \qquad |a_{ij}| \neq 0,$$

depends on nine independent homogeneous parameters, the nine a's. It is, therefore, an eight-parameter group. Accordingly, we denote it by G_8.

The group of affine transformations,

$$G_6 \qquad \begin{aligned} x' &= a_1x + b_1y + c_1, \\ y' &= a_2x + b_2y + c_2, \end{aligned} \qquad \Delta = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \neq 0.$$

is a subgroup of G_8 consisting of all the collineations which leave the line at infinity, $x_3 = 0$, fixed.

The group of rigid motions,

$$G_3 \qquad \begin{aligned} x' &= x\cos\theta - y\sin\theta + a, \\ y' &= x\sin\theta + y\cos\theta + b, \end{aligned}$$

is in turn a subgroup of G_6. It is made up of all the affine transformations which leave fixed each of the circular points at infinity and possess distance as an *absolute* invariant (Ch. VIII, § 7).

Projective geometry is the geometry of properties invariant with respect to the projective group G_8.

A property which is invariant with respect to the affine group G_6, but not with respect to G_8, is an affine property. Discussion of affine properties, with or without the adjunction of projective properties, constitutes affine geometry.

The properties which are invariant with respect to the metric group G_3, but not with respect to the affine group G_6, are metric properties.* The study of them, either by themselves or in conjunction with affine and projective properties, is the purpose of metric or Euclidean geometry.

In Ch. VIII, §§ 6, 7, we exploited the advantages of interpreting metric geometry as a "subgeometry" of projective geometry, obtained by demanding that two conjugate-imaginary points remain fixed. In the previous paragraphs we noted similar advantages enjoyed by affine geometry as "a projective geometry with a fixed line."

Both affine geometry and metric geometry are, however, *in themselves*, geometries of the finite plane, which employ nonhomogeneous in preference to homogeneous coordinates. In both, the principle of duality fails to function, and the point, as fundamental element, outweighs by far the straight line in importance.

EXERCISES

1. Classify the figures of Ch. II, § 1, Ex. 1 as projective, affine, or metric.

2. Do the same for the following figures and properties.

(*a*) A line-segment and its mid-point; (*b*) a pencil of intersecting straight lines; (*c*) a central conic; (*d*) an ellipse; (*e*) a central conic and its center; (*f*) a central conic and its foci; (*g*) the eccentricity of a conic.

* The introduction of affine geometry enables us to draw finer lines in classifying properties. Those properties which we previously designated as metric are here subdivided into (*a*) affine properties and (*b*) the properties which we now, and henceforth, designate as metric properties.

7. Geometries in the Complex Plane. In constructing a geometry, we are at liberty to specify whether we shall take the coefficients in the controlling group of transformations real or complex. We are also free to choose whether we shall study properties of real or complex configurations.

Thus far we have studied the properties of *real* configurations with respect to groups of *real* transformations. In Ch. VIII we discussed real conics, that is, conics represented by equations with real coefficients, with respect to the group of real rigid motions. In Ch. IX we treated real one-dimensional linear transformations with reference to changes of projective coordinates which were real.

If, in the case of the conics, we were to take the coefficients in the equation of the general conic, and those in the equations of the general transformation, complex instead of real, the classification of conics would be simpler. We should have merely the parabolas, tangent to the line at infinity, and the central conics, meeting the line at infinity in distinct points. There would no longer be a distinction between ellipses and hyperbolas; for, the expression $B^2 - 4AC$ would be complex and hence have no sign.

This case is typical. Complex geometry based on a group of complex transformations exhibits, in general, a smaller number of subdivisions than the corresponding real geometry based on the corresponding group of real transformations.

Complex geometry is, however, so abstract that we prefer to confine ourselves in the main to the consideration of real geometry with respect to groups of real transformations.

EXERCISE. Classify one-dimensional linear transformations with complex coefficients, and obtain a normal form for each type with respect to changes of projective coordinates with complex coefficients.

CHAPTER XII

POINT CONICS AND LINE CONICS

1. Point Conics. We shall henceforth write the general equation of the second degree in homogeneous point coordinates in the form

$$a_{11}x_1^2 + a_{22}x_2^2 + a_{33}x_3^2 + 2\,a_{23}x_2x_3 + 2\,a_{31}x_3x_1 + 2\,a_{12}x_1x_2 = 0,$$

and at the same time introduce a_{32}, a_{13}, a_{21} equal respectively to a_{23}, a_{31}, a_{12}:

$$a_{23} = a_{32}, \qquad a_{31} = a_{13}, \qquad a_{12} = a_{21}.$$

The equation can then be put in the form

$$\begin{aligned} &a_{11}x_1x_1 + a_{12}x_1x_2 + a_{13}x_1x_3 \\ + &a_{21}x_2x_1 + a_{22}x_2x_2 + a_{23}x_2x_3 \\ + &a_{31}x_3x_1 + a_{32}x_3x_2 + a_{33}x_3x_3 = 0, \end{aligned}$$

and hence written

$$\sum_{j=1}^{3} a_{1j}x_1x_j + \sum_{j=1}^{3} a_{2j}x_2x_j + \sum_{j=1}^{3} a_{3j}x_3x_j = 0,$$

or

$$\sum_{j=1}^{3} (a_{1j}x_1x_j + a_{2j}x_2x_j + a_{3j}x_3x_j) = 0,$$

or, still more compactly,

$$\sum_{j=1}^{3}\sum_{i=1}^{3} a_{ij}x_ix_j = 0.$$

One summation sign suffices to do the work of both. Thus we obtain the final condensed form

$$(1) \qquad \sum_{i,\,j=1}^{3} a_{ij}x_ix_j = 0, \qquad a_{ij} = a_{ji}.$$

Definition 1. *The locus of the points whose coordinates satisfy an equation of the form* (1), *where the coefficients are real and not all zero,** *is a real point conic.*

The determinant $|a_{ij}|$ is called the *discriminant* of the conic, and the matrix $\|a_{ij}\|$, the *matrix* of the conic.

* If the coefficients are all zero, the locus of (1) consists of all the points in the plane. Conversely, *if the locus of* (1) *is all the points of the plane, all the coefficients in* (1) *are zero.* For, if (1) is satisfied by the coordinates of all points (x_1, x_2, x_3), it is satisfied by (1, 0, 0), (0, 1, 0), (0, 0, 1), (0, 1, 1), (1, 0, 1), (1, 1, 0); hence we find that $a_{11} = a_{22} = a_{33} = a_{23} = a_{31} = a_{12} = 0$.

We assume that x_1, x_2, x_3 are projective coordinates which may be thought of as general, or specialized (affine or metric), according to the needs of any given problem.

Intersection of a Line and a Conic. If $r : (r_1, r_2, r_3)$ and $s : (s_1, s_2, s_3)$ are two distinct points on a straight line. an arbitrary point on the line has the coordinates

$$x = \lambda r + \mu s. \tag{2}$$

This point lies on the conic (1) if and only if *

$$\sum a_{ij}(\lambda r_i + \mu s_i)(\lambda r_j + \mu s_j) = 0,$$

or

$$\lambda^2 \sum a_{ij} r_i r_j + \lambda\mu(\sum a_{ij} r_i s_j + \sum a_{ij} s_i r_j) + \mu^2 \sum a_{ij} s_i s_j = 0. \tag{3}$$

Since $a_{ij} = a_{ji}$, $\sum a_{ij} s_i r_j$ is the same as $\sum a_{ji} r_j s_i$. But the latter sum is unchanged if we interchange i and j. Hence

$$\sum a_{ij} s_i r_j = \sum a_{ij} r_i s_j, \tag{4}$$

and (3) becomes

$$\lambda^2 \sum a_{ij} r_i r_j + 2\lambda\mu \sum a_{ij} r_i s_j + \mu^2 \sum a_{ij} s_i s_j = 0. \tag{5}$$

This is a homogeneous quadratic equation in λ, μ with constant coefficients. If the coefficients are not all zero, the equation determines two points $\lambda_1 r + \mu_1 s$, $\lambda_2 r + \mu_2 s$ common to the line and the conic. If the coefficients are all zero, every point (2) of the line lies on the conic.

THEOREM 1. *A straight line intersects a point conic in two points, distinct or coincident, or is entirely contained in the conic.*

By means of this theorem, we can prove

THEOREM 2. *If two point conics are identical, their equations are proportional.*†

* By Σ we shall always mean $\sum\limits_{i,j=1}^{3}$.

† *Proof.* Let

$$\alpha \equiv \Sigma a_{ij} x_i x_j = 0, \qquad \beta \equiv \Sigma b_{ij} x_i x_j = 0$$

have the same locus, and let r be a point not on this locus. Form the linear combination

$$\beta - k\,\alpha \equiv \Sigma b_{ij} x_i x_j - k\,\Sigma a_{ij} x_i x_j = 0,$$

choosing k so that the locus of $\beta - k\,\alpha = 0$ contains the point r:

$$k = (\Sigma b_{ij} r_i r_j)/(\Sigma a_{ij} r_i r_j).$$

Since r is not a point on the common locus of $\alpha = 0$ and $\beta = 0$, an arbi-

Definition 2. *If $\sum a_{ij}x_ix_j$ can be factored into two linear factors:*

$$\sum a_{ij}x_ix_j \equiv (c|x)(d|x),$$

the point conic (1) *is called degenerate.*

The conic then consists of the two straight lines $(c|x) = 0$, $(d|x) = 0$. Conversely, if the conic (1) consists of two straight lines $(c|x) = 0$, $(d|x) = 0$, equation (1) and the equation $(c|x)(d|x) = 0$ have the same locus and are, by Th. 2, proportional; hence

$$\sum a_{ij}x_ix_j \equiv k(c|x)(d|x), \qquad k \neq 0,$$

and the conic (1) is degenerate.

Theorem 3. *A point conic is degenerate if and only if it consists of two straight lines, distinct or coincident.*

If the lines are distinct, their point of intersection has the property that *every* line joining it to a second point of the conic is contained in the conic. If the lines are coincident, every point of the conic has this property.

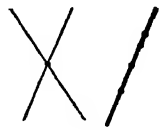


Fig. 1

Definition 3. *A point is called a singular point of a point conic if every line determined by it and a point of the conic is contained in the conic.*

We have noted that a degenerate conic always has at least one singular point. Conversely, if a conic has a singular point P, it must be constituted solely of straight lines through P* and hence is degenerate.

Theorem 4. *A nondegenerate point conic has no singular point. A degenerate point conic has one singular point, or all of its points are singular, according as it consists of two distinct, or of two coincident, lines.*

We give in the next theorem analytic conditions for a singular point.

trary line L through r intersects this locus in just two points. Then L intersects the locus $\beta - k\alpha = 0$ in at least three points—these two and r — and therefore, by Th. 1, is a part of the locus $\beta - k\alpha = 0$. But L was any line through r. Hence the locus $\beta - k\alpha = 0$ consists of all the points in the plane. Consequently, by Def. 1, footnote, $b_{ij} - k a_{ij} = 0$ or $b_{ij} = k a_{ij}$ for $i, j = 1, 2, 3$, and β is proportional to α; in fact, $\beta \equiv k\alpha$.

* The number of lines must be two, since otherwise a line not through P would intersect the conic in a finite number of points different from two and Theorem 1 would be contradicted. Of course the two lines may be distinct or coincident.

We postpone the proof for a moment in order not to break the thread of the present argument.

THEOREM 5. *The point r is a singular point of the point conic* (1) *if and only if r_1, r_2, r_3 satisfy the equations*

$$(6)\qquad \begin{aligned} a_{11}r_1 + a_{12}r_2 + a_{13}r_3 &= 0,\\ a_{21}r_1 + a_{22}r_2 + a_{23}r_3 &= 0,\\ a_{31}r_1 + a_{32}r_2 + a_{33}r_3 &= 0. \end{aligned}$$

Since these equations have a solution other than 0, 0, 0 if and only if $|a_{ij}| = 0$, there exists a singular point if and only if the discriminant of the conic is zero. Hence, we conclude

THEOREM 6. *A point conic is degenerate if and only if its discriminant vanishes.*

If the rank of the matrix of equations (6) is two, there is one singular point; if the rank is one, there are ∞^1 singular points, all the points of a line.

COROLLARY. *A degenerate point conic consists of two distinct, or two coincident, straight lines according as the rank of its matrix is two or one.*

Since equations (6) can be written

$$\sum_{j=1}^{3} a_{1j}r_j = 0, \qquad \sum_{j=1}^{3} a_{2j}r_j = 0, \qquad \sum_{j=1}^{3} a_{3j}r_j = 0,$$

they are equivalent to the identity

$$\Big(\sum_{j=1}^{3} a_{1j}r_j\Big)y_1 + \Big(\sum_{j=1}^{3} a_{2j}r_j\Big)y_2 + \Big(\sum_{j=1}^{3} a_{3j}r_j\Big)y_3 \equiv 0,$$

and hence to either of the identities

$$\sum a_{ij}y_ir_j \equiv 0, \qquad \sum a_{ij}r_iy_j \equiv 0.$$

Accordingly, we can restate Theorem 5 as follows.

THEOREM 5 (*Restated*). *The point r is a singular point of the point conic* (1) *if and only if*

$$(7)\qquad \sum a_{ij}r_iy_j \equiv 0.$$

In proving the theorem we begin by assuming that r is a singular point. Let y be *any* point of the plane other than r. A point of the line of r and y which lies on (1) has coordinates

$$x = r + \mu y,$$

where μ is a root of the equation

$$(8)\qquad \sum a_{ij}r_ir_j + 2\mu\sum a_{ij}r_iy_j + \mu^2\sum a_{ij}y_iy_j = 0.$$

Since r is a singular point, the line of r and y either is contained in the conic or meets the conic only in the point r, according as y is, or is not, on the conic. In the first case, all the coefficients in (8) vanish, and in the second, $\mu = 0$ is the only root of (8), so that the first two coefficients vanish. Hence, in either case,

$$(9) \qquad \sum a_{ij}r_ir_j = 0, \qquad \sum a_{ij}r_iy_j = 0, \qquad y \neq r.$$

The identity (7) is now seen to hold for all points y; the second of the equations (9) verifies it for any point y other than r and the first verifies it for the point r.

Suppose, conversely, that a point r is given for which the identity (7) is true. Then equations (9) hold regardless of the choice of the point y, and the first two coefficients in (8) are always zero. If y is taken, in particular, as *any* point on the conic, the third coefficient in (8) vanishes and the line of r and y is contained in the conic. Hence, by definition, the point r is a singular point.

THEOREM 7. *If a point conic contains a line, it is degenerate.*

This theorem, though intuitively obvious, is not easy to prove. We can reason that if the conic (1) contains the line $(c|x) = 0$, $\sum a_{ij}x_ix_j$ must contain $(c|x)$ as a factor; the second factor is then also linear and the conic is degenerate. There is a general theorem in higher algebra to support this reasoning; an appeal to it establishes the desired result.*

* *Direct Proof.* If the conic consists only of the points of the line c, it is identical with the conic $(c|x)^2 = 0$ and hence, by Th. 2, is degenerate.

If the conic contains a point s not on the line c, there exists a point y on c so that the line of y and s belongs to the conic. For, y lies on c if and only if

$$(10) \qquad (c|y) = 0,$$

and the line of s and y is contained in the conic if and only if all the coefficients in

$$\lambda^2 \Sigma a_{ij}s_is_j + 2\lambda\mu \Sigma a_{ij}s_iy_j + \mu^2 \Sigma a_{ij}y_iy_j = 0$$

vanish, or, since the first and last coefficients are by hypothesis zero, if and only if

$$(11) \qquad \Sigma a_{ij}s_iy_j = 0.$$

Equations (10) and (11), being linear and homogeneous in y_1, y_2, y_3, always have a solution other than 0, 0, 0. Thus our contention is proved.

The conic now contains two lines, the given line $(c|x) = 0$ and the line $(d|x) = 0$ of the points y and s. If it contained any other point, it would contain all the lines through this point, by Th. 1, and hence comprise all the points of the plane. Consequently, it consists precisely of the two lines, $(c|x)(d|x) = 0$, and is degenerate.

EXERCISES

1. By the method of the text find the points of intersection of the line joining the points (1, 4, 1), (5, 0, 1) with the conic

$$x_1x_2 - 6x_3^2 = 0.$$

2. Determine the points of intersection of the line

$$x_1 - 2x_2 + x_3 = 0$$

and the conic

$$x_1^2 + 2x_2^2 - x_3^2 - x_1x_2 - x_2x_3 = 0.$$

3. Verify the identity (4) by writing out and comparing the two members.

4. Ascertain the rank of the matrix of each of the following conics. If the conic is degenerate, find its singular points and the lines of which it consists.

(*a*) $$2x_1^2 - x_2^2 + 5x_3^2 - 4x_2x_3 + 7x_3x_1 - x_1x_2 = 0.$$

(*b*) $$2x_1^2 + 3x_2^2 - x_3^2 + 2x_2x_3 - x_3x_1 + x_1x_2 = 0.$$

(*c*) $$x_1^2 + 4x_2^2 + 4x_3^2 - 8x_2x_3 + 4x_3x_1 - 4x_1x_2 = 0.$$

5. Show that the definition of a point conic is independent of the system of projective coordinates chosen.

2. Nondegenerate Point Conics and Their Tangent Lines. From Theorems 1 and 7 of § 1 we conclude

THEOREM 1. *A straight line always intersects a nondegenerate point conic in two points, distinct or coincident.*

If the two points coincide in a single point, we say that the line meets the conic twice in this point.

THEOREM 2. *Through a point of a nondegenerate point conic there passes just one line which intersects the conic twice in the point.*

Let the conic be

(1) $$\sum a_{ij}x_ix_j = 0, \qquad a_{ij} = a_{ji}, \qquad |a_{ij}| \neq 0,$$

and let r be the given point on it: $\sum a_{ij}r_ir_j = 0$. If a point y moves always so that the line of y and r meets the conic twice in r, that is, so that $\lambda = 0$ is always a double root of the equation

$$\lambda^2 \sum a_{ij}y_iy_j + 2\lambda \sum a_{ij}y_ir_j + 0 = 0,$$

then

(2) $$\sum a_{ij}y_ir_j = 0,$$

and conversely. Equation (2) represents, therefore, the locus of the point y. It is a linear equation in y_1, y_2, y_3:

$$\left(\sum_{j=1}^{3} a_{1j}r_j\right)y_1 + \left(\sum_{j=1}^{3} a_{2j}r_j\right)y_2 + \left(\sum_{j=1}^{3} a_{3j}r_j\right)y_3 = 0,$$

not all of whose coefficients can be zero (Ex. 1), and therefore represents a straight line, which passes through the point r.

DEFINITION. *The unique line through a point of a nondegenerate point conic which meets the conic twice in the point is the tangent to the conic at the point.*

It is evident from the proof of Theorem 1 that the line (2) is the tangent to (1) at the point r.

THEOREM 2. *The equation of the tangent to the point conic* (1) *at the point* r *is*

$$(3)\qquad \sum a_{ij} r_i x_j = 0 \qquad \textit{or} \qquad \sum a_{ij} x_i r_j = 0.$$

It is to be noted that the equation is obtainable from (1) by replacing one set of x's in (1) by the corresponding r's.*

Condition that a Line be Tangent to a Nondegenerate Point Conic. The definition of a tangent implies that a line is tangent to the conic (1) if and only if the two points in which it intersects (1) are coincident. It is, then, a simple matter to deduce the condition that a line be tangent to (1) when the line is determined by two points (Ex. 6).

The problem is more interesting when the line is defined by its coordinates. Suppose that the line $u : (u_1, u_2, u_3)$ is tangent to (1). If the point of contact is r, coordinates of the line, as the tangent at r, are

$$\sum_{j=1}^{3} a_{1j} r_j, \qquad \sum_{j=1}^{3} a_{2j} r_j, \qquad \sum_{j=1}^{3} a_{3j} r_j.$$

Hence

$$(4)\qquad \sum_{j=1}^{3} a_{1j} r_j = \rho u_1, \qquad \sum_{j=1}^{3} a_{2j} r_j = \rho u_2, \qquad \sum_{j=1}^{3} a_{3j} r_j = \rho u_3.$$

Since r lies on u,

$$(5)\qquad u_1 r_1 + u_2 r_2 + u_3 r_3 = 0.$$

In (4) and (5) we have four homogeneous linear equations,

$$(6)\qquad \begin{aligned} a_{11} r_1 + a_{12} r_2 + a_{13} r_3 - u_1 \rho &= 0, \\ a_{21} r_1 + a_{22} r_2 + a_{23} r_3 - u_2 \rho &= 0, \\ a_{31} r_1 + a_{32} r_2 + a_{33} r_3 - u_3 \rho &= 0, \\ u_1 r_1 + u_2 r_2 + u_3 r_3 - 0\ \rho &= 0, \end{aligned}$$

* That the tangent to (1) at the point r, as we have here defined it, is the limiting position of the secant joining r to a neighboring point s of (1), as required by the general definition of a tangent, is readily proved.

which by hypothesis have a solution for r_1, r_2, r_3, ρ other than 0, 0, 0, 0. Consequently,

$$(7) \qquad \begin{vmatrix} a_{11} & a_{12} & a_{13} & u_1 \\ a_{21} & a_{22} & a_{23} & u_2 \\ a_{31} & a_{32} & a_{33} & u_3 \\ u_1 & u_2 & u_3 & 0 \end{vmatrix} = 0.$$

Given, conversely, a line u whose coordinates satisfy (7). Since the determinant in (7) is zero and $|a_{ij}| \neq 0$, the system of equations (6) is of rank three and has solutions other than 0, 0, 0, 0. These are the solutions of the first three equations. Since each two of them are proportional, it suffices to consider one of their number: r_1, r_2, r_3, ρ. If the r's in this solution were all zero, the first three equations would reduce to $u_1\rho = 0$, $u_2\rho = 0$, $u_3\rho = 0$ and ρ would be zero; vice versa, if ρ were zero, then, since $|a_{ij}| \neq 0$, the r's would all vanish. Hence, there exists a unique point (r_1, r_2, r_3) and a corresponding value of ρ, not zero, so that equations (4) and (5) are satisfied. Since (5) becomes, by virtue of (4),

$$r_1 \sum_{j=1}^{3} a_{1j}r_j + r_2 \sum_{j=1}^{3} a_{2j}r_j + r_3 \sum_{j=1}^{3} a_{3j}r_j = 0 \qquad \text{or} \qquad \sum a_{ij}r_ir_j = 0,$$

the point r is on the conic, and equations (4) then say that the line u is the tangent at r.

Equation (7) is a homogeneous quadratic equation in u_1, u_2, u_3. We find, on expanding the determinant, that it can be written in the form

$$(8) \qquad \sum A_{ij}u_iu_j = 0,$$

where A_{ij} is the cofactor of a_{ij} in $|a_{ij}|$ and (Ch. I, § 6, Ex. 3)

$$A_{ij} = A_{ji}, \qquad |A_{ij}| = |a_{ij}|^2.$$

THEOREM 3. *The line u is tangent to the nondegenerate point conic* (1) *if and only if u_1, u_2, u_3 satisfy equation* (7) *or the equivalent equation* (8).

Since equation (8) is satisfied by the coordinates of those and only those lines which are tangent to (1), it represents the totality of these tangents. This totality we shall call a *line curve*, in particular, the line curve which *corresponds* to (1). Inasmuch as (8) is of the second degree in u_1, u_2, u_3 and $|A_{ij}| \neq 0$, it is natural to designate this line curve as *a nondegenerate line conic.*

THEOREM 4. *The line curve corresponding to a nondegenerate point conic is a nondegenerate line conic. If the equation of the point conic is*

$$\sum a_{ij}x_ix_j = 0, \qquad a_{ij} = a_{ji}, \qquad |a_{ij}| \neq 0,$$

the equation of the line conic is

$$\sum A_{ij}u_iu_j = 0, \qquad A_{ij} = A_{ji}, \qquad |A_{ij}| \neq 0.$$

Before proceeding further it is essential that the student acquire the fundamentals of the theory of line curves, as set forth in Ch. XIII, §§ 1, 2.

EXERCISES

1. Show that equation (3) represents a line unless the conic is degenerate and r is a singular point.

2. Find the equation of the tangent at the point $(i, i, 1)$ to

$$x_1^2 + 2x_2^2 + 3x_3^2 + x_1x_3 - x_2x_3 = 0.$$

3. Find the equation of the line conic which corresponds to the given point conic.

(a) $x_1^2 + x_2^2 - x_3^2 = 0$; (b) $x_1^2 + x_2^2 + x_3^2 = 0$;

(c) $x_2^2 - 2x_1x_3 = 0$; (d) $x_1x_2 + x_2x_3 + x_3x_1 = 0$;

(e) $x_1^2 + x_2^2 + x_3^2 - 2x_2x_3 - 2x_3x_1 - 2x_1x_2 = 0$.

4. Show that equation (8) is equivalent to equation (7).

5. Is the line joining the points (2, 1, 3), (1, 1, 1) tangent to the conic of Ex. 2?

6. Prove that a condition necessary and sufficient that the line joining the points r and s be tangent to the conic (1) is that

$$(\Sigma a_{ij}r_ir_j)(\Sigma a_{ij}s_is_j) - (\Sigma a_{ij}r_is_j)^2 = 0.$$

7. Find the tangents to the conic

$$x_1^2 - 3x_2^2 - 4x_3^2 = 0$$

which pass through the point $(1, -2, 2)$.

Suggestion. Find the locus of the point y moving so that the line joining it to the given point is always tangent to the conic.

8. The preceding problem in the general case.

3. Line Conics. Since the theory of line conics is the dual of that of point conics, we content ourselves with a brief description of it.

DEFINITION 1. *The totality of lines whose coordinates satisfy an equation of the form*

$$\sum b_{ij}u_iu_j = 0, \qquad b_{ij} = b_{ji}, \tag{1}$$

where the coefficients are real and not all zero, is a real line conic.

The definition is independent of the system of projective coordinates chosen.

THEOREM 1. *Through a point pass just two lines of a line conic, or all the lines through the point belong to the conic.*

In line geometry a point consists of all the lines which pass through it. Accordingly, if all the lines through a point belong to a line conic, it is natural to say that the point belongs to the conic.

The treatment of degenerate line conics follows that of degenerate point conics. A line conic is degenerate if and only if it consists of two points, distinct or coincident. A line is a *singular line* of a line conic if *every* point determined by it and a line of the conic belongs to the conic. A degenerate line conic has one singular line or all of its lines are singular, according as it consists of two distinct, or two coincident, points. A nondegenerate line conic has no singular line.

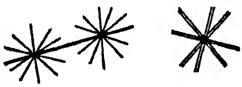


FIG. 2

THEOREM 2. *The line r is a singular line of* (1) *if and only if*

$$\sum b_{ij} r_i v_j \equiv 0, \qquad \textit{for all } v, \tag{2}$$

or

$$\sum_{j=1}^{3} b_{1j} r_j = 0, \qquad \sum_{j=1}^{3} b_{2j} r_j = 0, \qquad \sum_{j=1}^{3} b_{3j} r_j = 0. \tag{3}$$

THEOREM 3. *The line conic* (1) *is degenerate if and only if its discriminant $|b_{ij}|$ is zero. It then consists of two distinct, or two coincident, points according as the rank of its matrix $\|b_{ij}\|$ is two or one.*

Finally, if a line conic contains a point, it is necessarily degenerate.

Nondegenerate Line Conics and Their Contact Points.

THEOREM 4. *Common to a point and a nondegenerate line conic there are always two lines, distinct or coincident.*

THEOREM 5. *On a given line of a nondegenerate line conic there is a unique point which has the property that the lines common to it and the conic are coincident in the given line.*

Let the given line of (1) be $r : \sum b_{ij} r_i r_j = 0$. A point on r can be thought of as determined by r and a second line v. The lines $u = \lambda v + r$ common to the point and (1) are defined by the roots of the equation

$$\lambda^2 \sum b_{ij} v_i v_j + 2\lambda \sum b_{ij} v_i r_j + 0 = 0. \tag{4}$$

If the line v moves always so that these two lines coincide in r, that is, so that $\lambda = 0$ is always a double root of (4), then

$$\sum b_{ij}v_ir_j = 0, \tag{5}$$

and conversely. Hence the envelope of the line v is a point which lies on r.

DEFINITION 2. *The unique point on a line of a nondegenerate line conic which has the line twice in common with the conic is the contact point of the conic on the line.*

THEOREM 6. *The equation of the contact point of the line conic on the line r is*

$$\sum b_{ij}r_iu_j = 0 \qquad \text{or} \qquad \sum b_{ij}u_ir_j = 0. \tag{6}$$

The definition implies that a point is a contact point of (1) if and only if the two lines which it has in common with (1) are coincident.

THEOREM 7. *The point x is a contact point of the nondegenerate line conic*

$$\sum b_{ij}u_iu_j = 0, \qquad b_{ij} = b_{ji}, \qquad |b_{ij}| \neq 0, \tag{7}$$

if and only if x_1, x_2, x_3 satisfy the equation

$$\sum B_{ij}x_ix_j = 0, \qquad B_{ij} = B_{ji}, \qquad |B_{ij}| \neq 0, \tag{8}$$

where B_{ij} is the cofactor of b_{ij} in $|b_{ij}|$.

Since (8) is satisfied by the coordinates of those and only those points which are contact points of (7), it represents the point curve which corresponds to (7) in the sense of Ch. XIII, § 2.

THEOREM 8. *The point curve corresponding to a nondegenerate line conic is a nondegenerate point conic. If* (7) *is the equation of the line conic,* (8) *is that of the point conic.*

EXERCISES

1. Find the lines of $u_1^2 + u_2^2 - u_3^2 = 0$ which pass through the point of intersection of the lines $(1, -2, 0)$, $(1, -1, -1)$.

2. Find the lines of the conic

$$u_1^2 - 3u_2^2 + u_3^2 - 2u_1u_2 + 5u_1u_3 = 0$$

which pass through the origin.

3. Determine the contact point of $v^2 + 9u = 0$ on the line $(-1, -3)$.

4. Show that the line conic

$$2u_1^2 - u_2^2 - 2u_3^2 + 5u_2u_3 - u_3u_1 - 3u_1u_2 = 0$$

is degenerate and find the points of which it consists.

5. Find the equation of the point conic which corresponds to

$$u_1^2 + 4u_3^2 + 2u_2u_3 + 2u_1u_3 = 0.$$

6. Prove Theorem 7. **7.** Prove Theorem 2.

8. Deduce a condition that the point of intersection of the lines r and s be a contact point of the nondegenerate line conic (7).

4. Point Conics and Line Conics. We have seen (§ 2, Th. 4) that to the nondegenerate point conic

$$(1\,a) \qquad \sum a_{ij}x_ix_j = 0, \qquad a_{ij} = a_{ji}, \qquad |a_{ij}| \neq 0$$

corresponds the nondegenerate line conic

$$(1\,b) \qquad \sum A_{ij}u_iu_j = 0, \qquad A_{ij} = A_{ji}, \qquad |A_{ij}| \neq 0,$$

where A_{ij} is the cofactor of a_{ij} in $|a_{ij}|$, and (§ 3, Th. 8) that to the nondegenerate line conic

$$(2\,a) \qquad \sum b_{ij}u_iu_j = 0, \qquad b_{ij} = b_{ji}, \qquad |b_{ij}| \neq 0$$

corresponds the nondegenerate point conic

$$(2\,b) \qquad \sum B_{ij}x_ix_j = 0, \qquad B_{ij} = B_{ji}, \qquad |B_{ij}| \neq 0,$$

where B_{ij} is the cofactor of b_{ij} in $|b_{ij}|$.

Thus, there is a relationship between the nondegenerate point conics and the nondegenerate line conics; the tangents to a nondegenerate point conic constitute a nondegenerate line conic, and the contact points of a nondegenerate line conic form a nondegenerate point conic.

Is this relationship reciprocal? If C and C' are two conics of opposite types and C' corresponds to C, will C correspond to C'? For example, if C is a nondegenerate point conic and C' is the nondegenerate line conic consisting of the tangents to C, will the contact points of C' be the points of C?

That the question is to be answered affirmatively is guaranteed by Theorem 2 of Ch. XIII, § 2. A proof of the theorem in the present case is readily given. Consider the conics (1 a) and (1 b). We know that (1 b) corresponds to (1 a). Vice versa, (1 a) corresponds to (1 b). For, the point conic corresponding to (1 b) has the equation

$$\sum \alpha_{ij}x_ix_j = 0,$$

where α_{ij} is the cofactor of A_{ij} in $|A_{ij}|$, and consequently, since $\alpha_{ij} = |a_{ij}|\, a_{ij}$ (Ch. I, § 6, Ex. 2), is the conic (1 a).

THEOREM 1. *The nondegenerate point conics and the nondegenerate line conics correspond in pairs.*

If C and C' are respectively the point conic and the line conic of a pair, the tangent lines to C are the lines of C' and the contact points of C' are the points of C; C, with its tangent lines adjoined, and C', taken with its contact points, are identical.

THEOREM 2. *The nondegenerate point conics with their tangent lines and the nondegenerate line conics with their contact points are identical.*

We shall call these identical configurations simply *nondegenerate conics.* A nondegenerate conic consists, then, of both points and lines. It is at one and the same time a point locus and a line envelope and can be thought of as generated by a point and a line moving as a unit so that the point traces the constituent point curve, and the line, the constituent line curve.

A nondegenerate conic has two equations, one in point coordinates and one in line coordinates. If the conic is given by (1 a) as a point locus, its equation in line coordinates is (1 b). Or, if (2 a) defines the conic as a line envelope, the equation in point coordinates is (2 b).

The situation is entirely different when it comes to considering degenerate conics. A degenerate point conic consists of two lines and hence has no equation in line coordinates. Similarly, a degenerate line conic has no equation in point coordinates.

There are, then, three distinct types of conics: the nondegenerate conics, the degenerate point conics, and the degenerate line conics.

EXERCISES

1. Find the equation in line coordinates of

$$a_{11}x_1^2 + a_{22}x_2^2 + a_{33}x_3^2 = 0, \qquad a_{11}a_{22}a_{33} \neq 0.$$

Apply the result to

$$\frac{x^2}{a^2} \pm \frac{y^2}{b^2} = 1.$$

2. Find the equation in line coordinates of

$$a_{22}x_2^2 + 2\,a_{13}x_1x_3 = 0, \qquad a_{22}a_{13} \neq 0,$$

and apply the result to

$$y^2 = 2\,mx.$$

3. Find the equation in point coordinates of

$$a_1u_2u_3 + a_2u_3u_1 + a_3u_1u_2 = 0, \qquad a_1a_2a_3 \neq 0.$$

4. Show that the points and tangents of a (real) conic are real or conjugate-imaginary in pairs.

5. Prove that the tangents to a nondegenerate conic at conjugate-imaginary points are conjugate-imaginary. State the dual.

6. Show that the constituent points of a degenerate line conic are real or conjugate-imaginary.

7. Prove that if an imaginary line meets a point conic in two distinct points, the conjugate-imaginary line meets the conic in the two conjugate-imaginary points.

5. Tangents to a Degenerate Point Conic and Contact Points of a Degenerate Line Conic. If r is a nonsingular point of the point conic

$$\sum a_{ij}x_ix_j = 0, \qquad a_{ij} = a_{ji}, \tag{1}$$

the equation

$$\sum a_{ij}x_ir_j = 0 \tag{2}$$

always represents a straight line (§ 2, Ex. 1). This line we know to be the tangent to (1) at r if (1) is nondegenerate, and we agree to call it the tangent at r when (1) is degenerate.

The line is, in this case, that one of the constituent lines of the conic which contains r. For,

$$\sum a_{ij}x_ix_j \equiv 2\,(c\,|\,x)(d\,|\,x),$$

$$\sum a_{ij}x_ir_j \equiv (c\,|\,r)(d\,|\,x) + (d\,|\,r)(c\,|\,x).$$

Consequently, if we assume, for example, that r lies on the line c, equation (2) reduces to $(c\,|\,x) = 0$.

If r is a singular point of the degenerate point conic (1), the coefficients of x_1, x_2, x_3 in (2) all vanish. It is natural, here, to recognize two lines as tangents at r, namely the two constituent lines of the conic. If the conic consists of two distinct lines, there is only one singular point and the two tangents at it are distinct. When the lines of the conic coincide, every point is a singular point and the two tangents in each point coincide.

We can sum up by saying that a degenerate point conic has just two tangent lines, distinct or coincident, and that each of the two lines is tangent, at every one of its points, to the conic.

Exercise. Discuss the contact points of a degenerate line conic.

6. Complex Conics. If the coefficients in a homogeneous quadratic equation in x_1, x_2, x_3 or in u_1, u_2, u_3 are complex, the equation is said to represent a *complex* point conic or a *complex* line conic. If, in particular, the coefficients are all real or can all be made real by sup-

pression of a common factor, the complex conic is *real.* Otherwise it shall be called *imaginary.*

The contents of the present chapter, with the exception of Exs. 4–7, § 4, are valid for complex conics.

EXERCISES

1. Show that the point conic

$$2x_1^2 + ix_1x_2 + x_2^2 + (3-i)x_1x_3 - 2x_2x_3 + 2x_3^2 = 0$$

is degenerate and find its constituent lines.

2. Determine the tangents to the conic

$$x_1^2 + ix_1x_2 - x_3^2 = 0$$

which pass through the point $(1, i, 1)$.

3. Find the real points on each of the following conics:

$$(a)\quad x_1^2 + x_2^2 + x_3^2 = 0; \qquad (b)\quad x_2^2 - x_1x_3 + ix_2x_3 = 0;$$

$$(c)\quad x_2^2 - (1-i)x_1x_3 - ix_3^2 = 0;$$

$$(d)\quad x_2^2 - x_1x_3 + i(x_1 - x_3)(x_1 - 2x_3) = 0.$$

4. Show that a complex point conic is real if and only if it is identical with the conjugate-complex conic.

CHAPTER XIII

POINT CURVES AND LINE CURVES

1. Point Curves and Line Curves. Consider the circle whose center is at the origin and whose radius is a:

$$(1) \qquad x^2 + y^2 = a^2.$$

From the standpoint of line geometry, the circle must be thought of as generated by lines. The lines which it is natural to choose are the tangent lines. The circle is then called the *envelope* of its tangent lines, and an equation in the line coordinates (u, v) which is satisfied by the coordinates of these, and only these, lines is known as an *equation in line coordinates* of the circle.

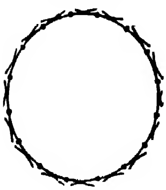


FIG. 1

The line (u, v) is tangent to (1) if and only if the distance from it to the origin is equal to a. Inasmuch as the equation in point coordinates (X, Y) of the line is (Ch. V, § 5)

$$(2) \qquad uX + vY + 1 = 0,$$

the distance in question is $1/\sqrt{u^2 + v^2}$. Hence the equation of the circle in line coordinates is

$$(3) \qquad u^2 + v^2 = \frac{1}{a^2} \cdot$$

Suppose, now, that an equation in u, v is given, for example,

$$(4) \qquad 4\,uv = 1.$$

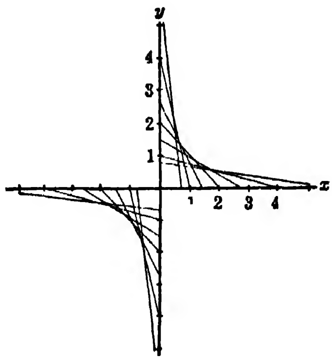


FIG. 2

This equation can be rewritten, in terms of the intercepts $-1/u$, $-1/v$ of the line (u, v), as $(-1/u)(-1/v) = 4$, and is therefore satisfied by the coordinates of those and only those lines the product of whose intercepts is equal to 4. We can draw with ease a large number of these lines. We thereby obtain, as the geometric representation of the equation

(Fig. 2), a clean cut picture of a curve, a curve defined by straight lines alone.

As another example, we take the equation

$$v = -4u^3. \tag{5}$$

Rewriting the equation in terms of the intercepts $-1/u$, $-1/v$ and plotting lines whose coordinates satisfy it, we find as its geometrical interpretation the curve shown in Fig. 3.

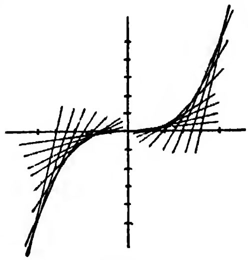


Fig. 3

In these two examples we have curves of a new type, curves defined by straight lines. We call them, for want of a better name, *line curves*, and, in distinction, call the curves defined by points, *point curves.*

As defined, point curves and line curves are as far apart as the poles. The former consist of points, the latter of lines. For example, equation (1) represents the point curve which consists of all the points at the distance a from the origin, and equation (3) represents the line curve which is made up of all the lines at the distance a from the origin. *A priori*, the two curves have nothing in common.

We know, however, that the tangents to the point circle (1) are precisely the lines of the line circle (3). We know, also, that on each line of (3) we can mark a point so that the resulting points are those of (1). Thus the point circle (1) and the line circle (3) are brought into relationship.

Figure 2 suggests that the lines of the line curve (4) are the tangents to a familiar point curve, a rectangular hyperbola with the axes as asymptotes. We can establish this relationship if we can choose the constant a in the equation

$$2xy = a \tag{6}$$

of a hyperbola of this type so that the product of the intercepts of each tangent to the hyperbola is equal to 4. The tangent to (6) at an arbitrary point (x, y) has the equation

$$yX + xY = a.$$

The product of the intercepts of the tangent is a^2/xy or $2a$, and will be equal to 4 if we take $a = 2$.

Thus, the tangents to the point hyperbola

$$xy = 1 \tag{7}$$

are the lines of the line curve (4). Vice versa, we can choose a point on each line of (4) so that these points are the points of (7). It is natural, then, to call the line curve (4) a line hyperbola.

EXERCISES

1. Reproduce on a larger scale than that of Fig. 3 the line curve represented by equation (5). Find the equation of the corresponding point curve, that is, the point curve whose tangents are the lines of the line curve.

2. Draw a graph representing the line curve $v^2 - u = 0$ and find the equation of the corresponding point curve.

3. Plot the line curve $u^2 + v^2 = u^2v^2$.

2. Continuation. Systematic Treatment. To round out the ideas concerning a line curve which are illustrated by the examples of the previous paragraph, we need a general theory by means of which points can be associated with a given line curve. It is natural to take as this theory the dual of the theory of tangent lines to a point curve. For the sake of comparison we shall arrange the two theories in parallel columns.

DEFINITION 1 *a*. *If S is the line joining a given point P to a neighboring point P′ of a point curve, the* TANGENT LINE *at P is the limit of S as P′ approaches P through points of the curve.*

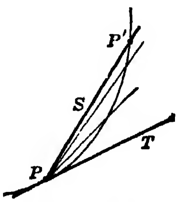


FIG. 4 *a*

THEOREM 1 *a*. *The equation of the tangent line to the point curve* *

(1 *a*) $$y = f(x)$$

DEFINITION 1 *b*. *If I is the point of intersection of a given line L and a neighboring line L′ of a line curve, the* CONTACT POINT *on L is the limit of I as L′ approaches L through lines of the curve.*

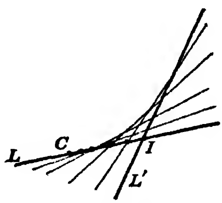


FIG. 4 *b*

THEOREM 1 *b*. *The equation of the contact point of the line curve*

(1 *b*) $$v = \phi(u)$$

* Since we are working in the complex plane, x and u are complex and $f(x)$ and $\phi(u)$ are functions of complex variables. We assume that these functions are single-valued functions which are *analytic*, that is, are continuous with continuous derivatives.

The reader who is not acquainted with analytic functions of a complex vari-

at the point (x, y) *is*

$$Y - y = \frac{dy}{dx}(X - x).$$

on the line (u, v) *is*

$$V - v = \frac{dv}{du}(U - u).$$

We prove Theorem 1 *b*. The equation of the point I in which the line $L : (u, v)$ and the neighboring line $L' : (u + \Delta u, v + \Delta v)$ intersect is

$$\begin{vmatrix} U & u & u + \Delta u \\ V & v & v + \Delta v \\ 1 & 1 & 1 \end{vmatrix} = 0, \quad \text{or} \quad \begin{vmatrix} U - u & u & \Delta u \\ V - v & v & \Delta v \\ 0 & 1 & 0 \end{vmatrix} = 0,$$

and hence can be written in the form

$$V - v = \frac{\Delta v}{\Delta u}(U - u).$$

When L' approaches L through lines of the curve, $\Delta v/\Delta u$ approaches dv/du. Hence the equation of the contact point is

$$V - v = \frac{dv}{du}(U - u).*$$

Definition 2 *a*. *The line curve which consists of the tangents to a given point curve shall be called the line curve corresponding to the given point curve.*

Definition 2 *b*. *The point curve which consists of the contact points of a given line curve shall be called the point curve corresponding to the given line curve.*

Theorems 1 *a* and 1 *b* make possible a general method for the determination of the curve of the one type which corresponds to a given curve of the other type. We illustrate the method by a pair of examples.

Let it be required to find the line curve which corresponds to the point curve

(2 *a*) $\quad 2xy = 1.$

Let it be required to find the point curve which corresponds to the line curve

(2 *b*) $\quad 2uv = 1.$

---

able may, if he wishes, consult Osgood, *Advanced Calculus*, Ch. XX. It suffices, however, for him to know that, as far as the content of the present chapter is concerned, he may work with these functions precisely as he works with functions of a real variable.

* A similar proof may be given for Theorem 1 *a*. The proofs, as well as the theorems, are projective, for they are valid regardless of whether (u, v) and (x, y) are thought of as general or specialized projective coordinates. The elementary proof of Theorem 1 *a* with which the reader is familiar assumes that (x, y) are metric coordinates and so establishes the validity of the theorem only for metric geometry.

The equation of the tangent to the point curve at the point (x, y) is

$$-yX - xY + 1 = 0.$$

Hence the coordinates (u, v) of the tangent are

$$u = -y, \qquad v = -x.$$

Eliminating x, y from these equations and $(2a)$, we obtain as the desired line curve

$$(2b) \qquad 2uv = 1.$$

To the point hyperbola $(2a)$ corresponds the line hyperbola $(2b)$.

The equation of the contact point of the line curve on the line (u, v) is

$$-vU - uV + 1 = 0.$$

Hence the coordinates (x, y) of the contact point are

$$x = -v, \qquad y = -u.$$

Eliminating u, v from these equations and $(2b)$, we obtain as the desired point curve

$$(2a) \qquad 2xy = 1.$$

To the line hyperbola $(2b)$ corresponds the point hyperbola $(2a)$.

We have here a point curve and a line curve each of which corresponds to the other. Thus, we have proved in a particular case the *fundamental* theorem:

THEOREM 2. *If one of two curves of opposite types corresponds to the other, the second corresponds to the first.*

In other words, if the tangents to a given point curve C are drawn, to form the corresponding line curve C', the contact points of C' will be the points of C; or, if the contact points of a given line curve C' are marked, to form the corresponding point curve C, the tangent lines of C will be the lines of C'. In both cases the relationship between the two curves is reciprocal: the tangent lines of C are the lines of C' and the contact points of C' are the points of C.

The theorem is by no means self-evident. A proof of it in the general case will be given in the next paragraph.

Suppose that equations $(1a)$ and $(1b)$ represent a point curve and a line curve which correspond. It is natural, then, to speak simply of *the curve.* We mean, thereby, both the point curve and the line curve. Accordingly, we may think of *the curve* as defined either by its points, or by its lines, as the occasion demands, and speak of $(1a)$ as its equation in point coordinates and of $(1b)$ as its equation in line coordinates.

For example, we think of the hyperbola of the foregoing problems as consisting of both its points and its tangents. In $(2a)$ we have its equation in point coordinates, and in $(2b)$, its equation in line coordinates.

The theory of corresponding point and line curves has exceptions which deserve special emphasis. A straight line, as a point curve, has but one tangent and hence cannot be considered as a line curve. Similarly, corresponding to the simplest of all line curves, a point, there is no point curve.

EXERCISES

1. Establish Theorem 1 *a*.

2. Prove that each of the two curves

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \qquad a^2u^2 + b^2v^2 = 1$$

corresponds to the other.

3. Find the equation in line coordinates of the curve whose equation in point coordinates is $y^2 = 2mx$.

4. Find the equation in point coordinates of the curve $v = 2u^2 - u + 1$.

3. Proof of the Fundamental Theorem. Let a point curve

$$(1) \qquad y = f(x),$$

not a straight line, be given. The equation of the tangent at the point (x, y) is

$$(2) \qquad y'X - Y + (y - y'x) = 0,$$

where $y' = dy/dx$. Hence the coordinates (u, v) of the tangent are

$$(3) \qquad u = \frac{y'}{y - y'x}, \qquad v = -\frac{1}{y - y'x}.$$

Elimination of x and y from equations (1) and (3) yields the equation of the line curve corresponding to the given point curve. The elimination is, however, unnecessary. Equations (3) constitute a parametric representation of the line curve in terms of x as the parameter; we need only think of y and y' as replaced by $f(x)$ and $f'(x)$.

To prove that the point curve corresponding to the line curve (3) is the given curve (1), it suffices to show that the contact point of (3) on the line (u, v) is the original point (x, y) of (1). The equation of this contact point is

$$(4) \qquad V - v = v'(U - u),$$

where $v' = dv/du$. Since, from (3),

$$(5) \qquad \frac{du}{dx} = \frac{y''y}{(y - y'x)^2}, \qquad \frac{dv}{dx} = -\frac{y''x}{(y - y'x)^2},$$

we have

(6) $$v' = -\frac{x}{y}.$$

Hence equation (4) becomes

(7) $$x\left(U - \frac{y'}{y - y'x}\right) + y\left(V + \frac{1}{y - y'x}\right) = 0,$$

and reduces immediately to the equation

$$xU + yV + 1 = 0$$

of the point (x, y).

In the same way it may be shown that, if a point curve corresponds to a line curve, the line curve corresponds to the point curve. Thus Theorem 2 of § 2 is established.

Critique. Equations (3) and subsequent equations are meaningless for a point (x, y) of (1) for which $y - y'x = 0$. This is natural, for the tangent (2) at (x, y) is then a line through (0, 0) and therefore has no coordinates (u, v); see Ch. V, § 5.

For a point P for which $y'' = 0$,* du/dx and dv/dx are zero and v' is undefined. It is natural, however, to define v' in this case as the limit approached by the ratio of dv/dx and du/dx when a point on the curve neighboring to P approaches P as a limit. Then (6) is valid for P.

The difficulty with (6) arising from the fact that y may be zero is superficial. We can write, instead of (4),

$$\frac{dv}{dx}(U - u) - \frac{du}{dx}(V - v) = 0.$$

Substitution of the values of the derivatives from (5) then leads directly to (7).

Our proposition is therefore established except for points of (1) at which the tangents go through (0, 0). That it holds also for these points is immediately realized when one imagines the origin properly shifted and the proof repeated.

EXERCISES

1. A curve has in point coordinates the parametric representation $x = t^2 - 1$, $y = t$. Find the corresponding parametric representation in line coordinates.

* If $y'' \equiv 0$, then $y = ax + b$ and (1) is a straight line, contrary to hypothesis.

2. Find a parametric representation in point coordinates for the curve

$$u = -\frac{1}{a}\sec\theta, \qquad v = \frac{1}{b}\tan\theta,$$

and identify the curve.

3. The four-cusped hypocycloid $x^{2/3} + y^{2/3} = a^{2/3}$ has the parametric representation

$$x = a\cos^3\theta, \qquad y = a\sin^3\theta.$$

Show that the corresponding parametric representation in line coordinates is

$$u = -\frac{1}{a}\sec\theta, \qquad v = -\frac{1}{a}\csc\theta$$

and hence find the equation in line coordinates.

4. One-Parameter Families of Lines. Envelopes. We have developed the theory of one-parameter families of lines from the point of view of line geometry. From this point of view, they are simply line curves.

The usual approach to one-parameter families of lines is from the opposite point of view. The family of lines is defined by an equation in point coordinates, and the problem is to find, not the equation of the line curve which the lines constitute, but the equation of the *envelope*, the point curve which has the lines as its tangents. The envelope is then the point curve, and the term "envelope," like the term "locus," belongs to point geometry.*

A one-parameter family of lines is represented by an equation in point coordinates of the type

$$f_1(t)\,x_1 + f_2(t)\,x_2 + f_3(t)\,x_3 = 0. \tag{1}$$

We assume that $f_1(t), f_2(t), f_3(t)$ are single-valued, analytic functions of the complex parameter t, and for our immediate purposes we rewrite the equation in nonhomogeneous form †

$$f(t)\,x + \phi(t)\,y + 1 = 0. \tag{2}$$

We shall describe three methods for finding the envelope of the family of lines (1) or (2).

The Method of Line Geometry. A parametric representation of the line curve consisting of the lines (2) is evidently

$$u = f(t), \qquad v = \phi(t). \tag{3}$$

* There are, unfortunately, no corresponding terms in line geometry.

† Here $f(t) = f_1(t)/f_3(t)$, $\phi(t) = f_2(t)/f_3(t)$. In dividing by $f_3(t)$ we are excluding the lines of the family which go through the point (0, 0). Of course, all the lines may go through this point; but the envelope is then just the point.

Elimination of t from these equations yields the equation in line coordinates of the envelope.

The envelope itself is the point curve corresponding to the line curve (3). The contact point (x, y) of (3) on the line (u, v) has the equation

$$\phi'(t)[U - f(t)] - f'(t)[V - \phi(t)] = 0,$$

or

$$-\phi' U + f' V + (f\phi' - f'\phi) = 0.$$

Hence the equations

$$(4) \qquad x = -\frac{\phi'}{f\phi' - f'\phi}, \qquad y = \frac{f'}{f\phi' - f'\phi}$$

constitute a parametric representation of the envelope and elimination of t from them gives the equation of the envelope.*

The Method of Analysis. According to this method,† the envelope is found by eliminating t from the equations

$$f(t)\, x + \phi(t)\, y + 1 = 0,$$
$$f'(t)\, x + \phi'(t)\, y \qquad = 0,$$

the second of which is the result of differentiating the first partially with respect to t. These equations are equivalent to (4), for the result of solving them for x and y is precisely (4).‡

The Method of Algebraic Geometry. We assume now that our family

* If the family consists of a pencil of lines, the envelope is a point and has no equation in point coordinates. If the point is finite, the expressions for x and y given in (4) are constants. If the point is at infinity, $f\phi' - f'\phi \equiv 0$; conversely, it can be shown that, if $f\phi' - f'\phi \equiv 0$, the lines of the family are all parallel.

† See Osgood, *Advanced Calculus*, Ch. VIII.

‡ The method of analysis can equally well be applied to equation (1). In this case we have

$$f_1(t)x_1 + f_2(t)x_2 + f_3(t)x_3 = 0,$$
$$f_1'(t)x_1 + f_2'(t)x_2 + f_3'(t)x_3 = 0.$$

Hence a parametric representation of the envelope is

$$(a) \qquad x_1 = f_2f_3' - f_3f_2', \qquad x_2 = f_3f_1' - f_1f_3', \qquad x_3 = f_1f_2' - f_2f_1'.$$

The corresponding representation of the given lines is

$$(b) \qquad u_1 = f_1(t), \qquad u_2 = f_2(t), \qquad u_3 = f_3(t).$$

The method of line geometry can also be applied to (1). Here we start with the equations (b) and show directly that the corresponding point curve is represented by equations (a).

of lines can be represented by an equation (1) in which $f_1(t)$, $f_2(t)$, $f_3(t)$ are polynomials in t. The equation can then be rewritten in the form

(5) $$\alpha_0 t^n + \alpha_1 t^{n-1} + \cdot \cdot \cdot \cdot \cdot \cdot + \alpha_{n-1} t + \alpha_n = 0,$$

where

$$\alpha_i \equiv a_{i1}x_1 + a_{i2}x_2 + a_{i3}x_3, \quad (i = 0, 1, 2, \cdots, n).$$

If $n = 1$, equation (5) becomes

$$\alpha_0 t + \alpha_1 = 0$$

and represents a pencil of lines, provided the equations $\alpha_0 = 0$, $\alpha_1 = 0$ represent straight lines which are distinct.

Consider, next, the quadratic family of lines

(6) $$\alpha_0 t^2 + 2\alpha_1 t + \alpha_2 = 0,$$

where $\alpha_0 = 0$, $\alpha_1 = 0$, $\alpha_2 = 0$ represent lines which are not concurrent. For a given point (x_1, x_2, x_3), equation (6) is in general quadratic in t. Hence there are in general two lines of the family which pass through a given point of the plane. Accordingly, it is reasonable to expect the envelope to be a point conic. If this is the case, the points of the envelope are the points for which the corresponding two lines of the family coincide, and the equation of the envelope is obtained by setting the discriminant of the quadratic equation (6) equal to zero:

(7) $$\alpha_1^2 - \alpha_0 \alpha_2 = 0.$$

The method is readily justified. Eliminating t from (6) and the equation obtained by differentiating (6) partially with respect to t, we obtain (7).

In the case of a cubic family of lines,

(8) $$\alpha_0 t^3 + \alpha_1 t^2 + \alpha_2 t + \alpha_3 = 0,$$

there are in general three lines through each point. The points at which at least two of these lines coincide are the points for which (8) has a multiple root, that is, the points for which the discriminant of (8) vanishes. These points actually constitute the envelope. For we know from algebra that (8) has a multiple root if and only if it and its derived equation,

(9) $$3\alpha_0 t^2 + 2\alpha_1 t + \alpha_2 = 0,$$

have a common root, and that the result of eliminating t from (8) and (9) is the discriminant of (8) equated to zero. But, by the method of analysis, the result of this elimination is the equation of the envelope.

Appealing to other sources for the discriminant of (8), we obtain as the equation of the envelope

$$(10) \qquad 27\,\alpha_0^2\alpha_3^2 - \alpha_1^2\alpha_2^2 + 4\,\alpha_0\alpha_2^3 + 4\,\alpha_1^3\alpha_3 - 18\,\alpha_0\alpha_1\alpha_2\alpha_3 = 0.$$

The extension of the method to the general case of the family defined by equation (5) is self-evident.

EXERCISES

In each of the following examples find the envelope of the line which moves subject to the conditions stated, obtaining its equation in line coordinates as well as its equation in point coordinates.

1. The sum of the intercepts on the axes is constant. First draw the graph, taking the constant equal to 4.

2. The difference of the intercepts, taken in a definite order, is constant.

3. The line always forms with the coordinate axes a triangle of given area.

4. The sum of the intercepts is equal to their product.

5. Two mutually perpendicular lines cut from the line a segment of given length.

6. Find the envelope of each of the following families of lines:

$$(a)\ x\,t^2 - 2\,y\,t + 3 = 0; \qquad (b)\ \ x \cot t + y \tan t + 4 = 0;$$
$$(c)\ \ t^3 + 3\,x\,t^2 + 3\,y\,t + 1 = 0.$$

7. Show that the equation in line coordinates of the circle with radius r and center in the point (a, b) is

$$(a\,u + b\,v + 1)^2 = r^2(u^2 + v^2).$$

In the following problems find first the equation in line coordinates of the envelope.

8. The sum of the squares of the distances from the line to two fixed points is constant.

9. The previous exercise, if the difference of the squares, in a definite order, is assumed given.

10. The product of the algebraic distances from the line to two fixed points is constant. Distinguish two cases, according as the two fixed points lie on the same, or opposite, sides of the line.

11. The preceding problem, if the sum of the two algebraic distances is taken as constant.

5. Algebraic Curves. *Point Curves.* The locus of the points whose coordinates satisfy an equation of the form

$$(1) \qquad f(x_1, x_2, x_3) = 0,$$

where $f(x_1, x_2, x_3)$ is a homogeneous polynomial in x_1, x_2, x_3 with real coefficients, not all zero, is a *real algebraic point curve.* The degree n

of the polynomial is called the *order* of the curve.

The algebraic point curves of orders one and two are the straight lines and point conics. An algebraic point curve of order three may have its equation written in the form

$$\sum_{i=1}^{3} \sum_{j=1}^{3} \sum_{k=1}^{3} a_{ijk} x_i x_j x_k = 0, \tag{2}$$

where each two a's with the same subscripts are equal.

We shall assume that the point curve (1) is *irreducible*, that is, that the polynomial $f(x_1, x_2, x_3)$ cannot be written as the product of two polynomials, neither of which is a constant.

THEOREM 1 *a*. *An algebraic point curve of order n is met in n points by every line.*

If r and s are two distinct points of the line, the points $x = \lambda r + \mu s$ in which the line meets the curve (1) are the points for which λ, μ satisfy the equation

$$f(\lambda r_1 + \mu s_1, \lambda r_2 + \mu s_2, \lambda r_3 + \mu s_3) = 0. \tag{3}$$

Since $f(x_1, x_2, x_3)$ is of degree n in x_1, x_2, x_3, the left-hand side of this equation is a homogeneous polynomial in λ, μ of degree n.* If all the coefficients in this polynomial were zero, the curve would contain the line and hence be reducible. Therefore the line intersects the curve in n points.

If r is a point of (1), $\lambda = 1$, $\mu = 0$ is a solution of (3); in other words, μ is a factor of the polynomial in (3). It is conceivable that a higher power of μ than the first, say μ^k, is a factor of the polynomial. Then k of the n points in which the line meets (1) coincide in r and we say that *the line meets* (1) *k times in r.*

What is the locus of a point y which moves always so that the line determined by it and a given point r of (1) *meets* (1) *at least twice in r?*

The points $x = r + \mu y$ in which the line of r and y meets (1) are those for which μ satisfies the polynomial equation

$$F(\mu) \equiv f(r_1 + \mu y_1, r_2 + \mu y_2, r_3 + \mu y_3) = 0. \tag{4}$$

The polynomial $F(\mu)$ is precisely the Maclaurin development for $F(\mu)$:

$$F(\mu) \equiv F(0) + F'(0)\mu + F''(0)\frac{\mu^2}{2!} + \cdots\cdots + F^{[n]}(0)\frac{\mu^n}{n!}\,.$$

* For example, for the cubic curve (2), equation (3) is

$$\lambda^3 \sum_{ijk}^{1-3} a_{ijk} r_i r_j r_k + 3\lambda^2\mu \sum_{ijk}^{1-3} a_{ijk} r_i r_j s_k + 3\lambda\mu^2 \sum_{ijk}^{1-3} a_{ijk} r_i s_j s_k + \mu^3 \sum_{ijk}^{1-3} a_{ijk} s_i s_j s_k = 0.$$

From (4),

$$F(\mu) = f(x_1, x_2, x_3), \qquad F'(\mu) = \sum_{i=1}^{3} f_i(x_1, x_2, x_3) y_i,$$

$$F''(\mu) = \sum_{ij}^{1-3} f_{ij}(x_1, x_2, x_3) y_i y_j, \qquad F'''(\mu) = \sum_{ijk}^{1-3} f_{ijk}(x_1, x_2, x_3) y_i y_j y_k,$$

where

$$x_1 = r_1 + \mu\, y_1, \quad x_2 = r_2 + \mu\, y_2, \quad x_3 = r_3 + \mu\, y_3,$$

and $f_i(x_1, x_2, x_3)$ denotes the first partial derivative of $f(x_1, x_2, x_3)$ with respect to x_i, $f_{ij}(x_1, x_2, x_3)$ the second partial derivative of $f(x_1, x_2, x_3)$ with respect to x_i and x_j, and so forth. Hence

$$F(0) = f(r_1, r_2, r_3), \qquad F'(0) = \sum_{i=1}^{3} f_i(r_1, r_2, r_3) y_i,$$

$$F''(0) = \sum_{ij}^{1-3} f_{ij}(r_1, r_2, r_3) y_i y_j, \qquad F'''(0) = \sum_{ijk}^{1-3} f_{ijk}(r_1, r_2, r_3) y_i y_j y_k.$$

By hypothesis, $f(r_1, r_2, r_3) = 0$. Thus the equation $F(\mu) = 0$ becomes

$$(5)\quad 0 = \mu \sum f_i(r_1, r_2, r_3) y_i + \frac{\mu^2}{2!} \sum f_{ij}(r_1, r_2, r_3) y_i y_j + \cdots + \frac{\mu^n}{n!} F^{[n]}(0),$$

and has $\mu = 0$ as a root at least twice if and only if y_1, y_2, y_3 satisfy

$$(6)\qquad f_1(r_1, r_2, r_3)\, y_1 + f_2(r_1, r_2, r_3)\, y_2 + f_3(r_1, r_2, r_3)\, y_3 = 0.$$

This, then, is the equation of our locus.

If the coefficients in (6) are all zero, the locus consists of all the points of the plane. In other words: *Every line through a point r of* (1) *for which*

$$(7)\qquad f_1(r_1, r_2, r_3) = 0, \qquad f_2(r_1, r_2, r_3) = 0, \qquad f_3(r_1, r_2, r_3) = 0$$

meets (1) *at least twice in r.* A point of an algebraic point curve with this property is known as a *singular point* of the curve.

If r is not a singular point, equation (6) represents a straight line which goes through the point r.* Thus, *there is just one line through a nonsingular point of an algebraic point curve which meets the curve at least twice in the point.* This line we call the *tangent* to the curve at the point.†

* Since the locus is that of all points which determine with r lines meeting (1) at least twice in r, it follows that all points on a line through r belong to the locus if one does. Hence the locus must in any case consist of lines through r.

† We leave it to the reader to show that this special definition of a tangent conforms to the general definition of § 2.

THEOREM 2 *a*. *The equation of the tangent to the point curve* (1) *at the nonsingular point r is*

$$f_1(r_1, r_2, r_3)x_1 + f_2(r_1, r_2, r_3)x_2 + f_3(r_1, r_2, r_3)x_3 = 0. \tag{8}$$

Line Curves. A real algebraic line curve is the totality of lines whose coordinates satisfy an equation of the form

$$\phi(u_1, u_2, u_3) = 0, \tag{9}$$

where $\phi(u_1, u_2, u_3)$ is a homogeneous polynomial in u_1, u_2, u_3 with real coefficients, not all zero. The degree m of the polynomial is called the *class* of the curve.

THEOREM 1 *b*. *An irreducible algebraic line curve of class m has m lines in common with every point.*

If every point on a line L of the line curve has the property that, among the lines common to it and the curve, the line L counts at least twice, the line L is called a *singular line*. The line r of (9) is singular if and only if

$$\phi_1(r_1, r_2, r_3) = 0, \qquad \phi_2(r_1, r_2, r_3) = 0, \qquad \phi_3(r_1, r_2, r_3) = 0. \tag{10}$$

There is a unique point on a nonsingular line of the line curve which has the property just described. It is defined as the *contact point* of the curve on the line.

THEOREM 2 *b*. *The contact point of* (9) *on the nonsingular line r has the equation*

$$\phi_1(r_1, r_2, r_3)u_1 + \phi_2(r_1, r_2, r_3)u_2 + \phi_3(r_1, r_2, r_3)u_3 = 0. \tag{11}$$

Corresponding Point and Line Curves. If the line u is the tangent to the point curve (1) at the nonsingular point x, then

$$\rho u_1 = f_1(x_1, x_2, x_3), \qquad \rho u_2 = f_2(x_1, x_2, x_3), \qquad \rho u_3 = f_3(x_1, x_2, x_3), \tag{12}$$

$$u_1x_1 + u_2x_2 + u_3x_3 = 0. \tag{13}$$

Suppose, conversely, that a line u is given for which the point x exists so that the x's and u's satisfy (12) and (13). Substituting the values of u_1, u_2, u_3 from (12) into (13), we have

$$x_1f_1(x_1, x_2, x_3) + x_2f_2(x_1, x_2, x_3) + x_3f_3(x_1, x_2, x_3) = 0. \tag{14}$$

By Euler's Theorem *

$$x_1f_1(x_1, x_2, x_3) + x_2f_2(x_1, x_2, x_3) + x_3f_3(x_1, x_2, x_3) = nf(x_1, x_2, x_3). \tag{15}$$

* See Osgood, *Advanced Calculus*, p. 122.

Hence

$$f(x_1, x_2, x_3) = 0,$$

and the point x lies on (1). Equations (12) then say that this point x is a nonsingular point of (1) and that the line u is the tangent at it.

THEOREM 3 *a*. *A necessary and sufficient condition that the line u be a tangent to the point curve* (1) *is that the point x exist so that the u's and x's satisfy equations* (12) *and* (13).

Suppose that we eliminate x_1, x_2, x_3 from (12) and (13). The result is an equation in u_1, u_2, u_3 which, since (12) and (13) are algebraic equations, can be reduced to the form

$$\phi(u_1, u_2, u_3) = 0, \tag{16}$$

where ϕ is a homogeneous polynomial in u_1, u_2, u_3.

Equation (16) represents the line curve corresponding to the point curve (1). For, it follows from Th. 3 *a* that the coordinates of the tangents at the nonsingular points of (1) satisfy (16), and it is a fact, the proof of which need not concern us here, that the lines whose coordinates satisfy (16) are, with certain well-warranted exceptions,* precisely these tangents.

THEOREM 4 *a*. *The line curve corresponding to the algebraic point curve* (1) *is algebraic. Its equation is obtained by eliminating x_1, x_2, x_3 from equations* (12) *and* (13).

The duals of Theorems 3 *a* and 4 *a* we leave to the reader.

If an algebraic point curve and an algebraic line curve

$$f(x_1, x_2, x_3) = 0, \qquad \phi(u_1, u_2, u_3) = 0 \tag{17}$$

are so related that one corresponds to the other, then, by our fundamental theorem, the second corresponds to the first. We call the two curves, taken together, an *algebraic curve* and refer to equations (17) as its equations in point coordinates and line coordinates, respectively. The degree of the first equation is the order of the algebraic curve, and that of the second, the class.

We state without proof that, if an algebraic curve of order n has no singular points, its class is

$$m = n(n - 1). \tag{18}$$

If singular points are present, the class is reduced. For example, a nondegenerate conic is of order 2 and has no singular points; hence

* The tangents to (1) at the singular points of (1); see § 6.

by (18) its class should be 2, as is actually the case. Again, an algebraic curve of order 3 without singular points is of class 6; if the curve has singular points, its class is either 4 or 3; see § 6.

Example. For the curve of the third order *

$$x_1^3 + x_2^3 + x_3^3 = 0,$$

$$f_1 = 3\,x_1^2, \qquad f_2 = 3\,x_2^2, \qquad f_3 = 3\,x_3^2.$$

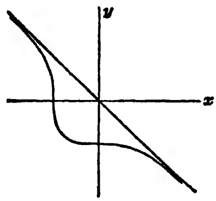


FIG. 5

Since these partial derivatives vanish simultaneously only when $x_1 = 0$, $x_2 = 0$, $x_3 = 0$, the curve has no singular points. Its class is, therefore, 6.

Equations (12) become here, on taking $\rho = 3$,

$$x_1^2 = u_1, \qquad x_2^2 = u_2, \qquad x_3^2 = u_3.$$

Substituting the values of x_1, x_2, x_3 obtained from these equations into (13), we find as the equation of the curve in line coordinates

$$\pm\, u_1^{3/2} \pm u_2^{3/2} \pm u_3^{3/2} = 0,$$

or, in rational form,

$$u_1^6 + u_2^6 + u_3^6 - 2\,u_2^3u_3^3 - 2\,u_3^3u_1^3 - 2\,u_1^3u_2^3 = 0.$$

Nonhomogeneous Equations of Algebraic Curves. The equation of the curve (1) in nonhomogeneous coordinates is

(19) $$F(x, y) = 0,$$

where

(20) $$f(x_1, x_2, x_3) \equiv x_3^n\, F(x_1/x_3, x_2/x_3).$$

We exclude the points of the curve on $x_3 = 0$.

Since, from (20),

$$f_1 = x_3^{n-1}\, F_x, \qquad f_2 = x_3^{n-1}\, F_y,$$

$$f_3 = n\, x_3^{n-1}\, F(x, y) - x_3^{n-2}\,(x_1 F_x + x_2 F_y),$$

the singular points of (19) are the points of (19) for which

$$F_x(x, y) = 0, \qquad F_y(x, y) = 0.$$

At a nonsingular point,

(21) $$y' = \frac{dy}{dx} = -\frac{F_x(x, y)}{F_y(x, y)}\,.$$

* Fig. 5 shows the curve in the metric plane.

If (x_0, y_0) is a nonsingular point of (19), not both $F_x(x_0, y_0)$ and $F_y(x_0, y_0)$ are zero. Assume, for example, that $F_y(x_0, y_0) \neq 0$.* The equation $F(x, y) = 0$ then defines y as a function of x,

$$y = f(x), \tag{22}$$

in a neighborhood of (x_0, y_0). This function is single-valued and analytic and its first derivative is given by (21), where it is understood that the y in $F_x(x, y)$ and $F_y(x, y)$ is the y of (22).†

Thus, a piece of the curve (19) in the neighborhood of a nonsingular point can be represented by an equation of the form employed in the opening paragraphs of the chapter. Hence our previous developments are perfectly general, applying as well to the case of a curve represented by an equation defining y implicitly as a function of x as in the simpler case where y is defined explicitly as a function of x. In particular, it follows that our proof of the fundamental theorem is as valid in the one case as in the other.

EXERCISES

In each of the following exercises, find the equation of the given curve in the opposite kind of coordinates. Give the order and the class of the curve.

1. $x_2^2 x_3 = x_1^3$.
2. $4u_1^3 + 27u_2u_3^2 = 0$.
3. $x^4 = y^3$.
4. $v = u^4$.
5. Apply the results of this section to point conics and line conics, showing that they reduce to those of Ch. XII.

6. Singular Points and Singular Lines of Algebraic Curves. Suppose that the algebraic curve

$$f(x_1, x_2, x_3) = 0, \tag{1}$$

has the singular point r:

$$f_1(r_1, r_2, r_3) = 0, \qquad f_2(r_1, r_2, r_3) = 0, \qquad f_3(r_1, r_2, r_3) = 0.\ddagger \tag{2}$$

The points $x = r + \mu y$ in which the line of r and an arbitrary point y meets the curve are, according to § 5, the points for which μ is a root of the equation

$$0 = \frac{\mu^2}{2!}\sum f_{ij}(r_1, r_2, r_3)y_iy_j + \frac{\mu^3}{3!}\sum f_{ijk}(r_1, r_2, r_3)y_iy_jy_k + \cdots + \frac{\mu^n}{n!}F^{[n]}(0).$$

* If $F_y(x_0, y_0) = 0$, think of x and y as interchanged in the argument which follows.

† This is the Implicit Function Theorem of Analysis; see Osgood, *Lehrbuch der Funktionentheorie*, 5th edition, p. 421; *Advanced Calculus*, p. 132.

‡ It is not necessary to add the equation $f(r_1, r_2, r_3) = 0$, for this equation follows from (2) by virtue of Euler's Theorem.

Hence the locus of the points y such that the line meets the curve at least *three* times in r has the equation

(3) $$\sum f_{ij}(r_1, r_2, r_3)y_iy_j = 0.$$

In case the coefficients in this equation are not all zero, we shall call the point r a *simple* singular point. The locus (3) is then a conic which, by the nature of our problem, consists of two straight lines through r.*

There are just two lines through a simple singular point of an algebraic curve which meet the curve at least three times in the point. These two lines are known as the *tangents* to the curve at the point.

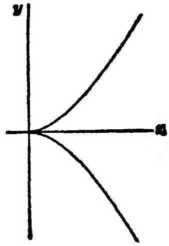


FIG. 6

According as the two tangents at a simple singular point are distinct or coincident, the point is called a *double point* or a *cusp*. The familiar curve of Fig. 6,

(4) $$x_1^3 - x_2^2x_3 = 0 \qquad (y^2 = x^3),$$

has a cusp at the point (0, 0, 1), whereas the curve of Fig. 7,

(5) $$x_1^3 + x_1^2x_3 - x_2^2x_3 = 0 \qquad (y^2 = x^3 + x^2),$$

has a double point at (0, 0, 1).

If the values of the second partial derivatives of $f(x_1, x_2, x_3)$ for the point r—the coefficients in (3)—are all zero, but the values of the third partial derivatives for r are not all zero, every line through r will meet the curve at least three times in r and there will be just three lines which meet it four times in r. These three lines, which are of course not necessarily distinct, we call the tangents at r. In general, if the partial derivatives of $f(x_1, x_2, x_3)$ of orders $1, 2, \cdots, k - 1$, where $k \geqq 2$, are all zero for the point r, but the kth partial derivatives do not all vanish for r, there will be k lines which we can justly call tangents at r. The point r is then known as a *k-fold singular point*, or *a singular point of multiplicity k*.

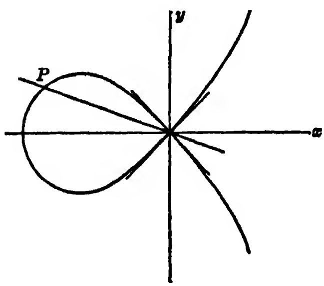


FIG. 7

The facts concerning singular lines are similar. A line r is a k-fold

* See the first footnote on p. 214.

singular line of $\phi(u_1, u_2, u_3) = 0$ if the partial derivatives of $\phi(u_1, u_2, u_3)$ of orders $1, 2, \cdots, k-1$, where $k \geqq 2$, all vanish for r, whereas the kth partial derivatives are not all zero for r. Every point of the line r has, then, the property that among the lines common to it and the curve the line r counts at least k times, and there are just k points on r each of which has the property that it has the line r at least $k+1$ times in common with the curve. These k points are called the *contact points* of the curve on the singular line.

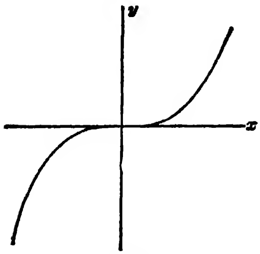


FIG. 8

We are primarily interested in the case $k = 2$ of a *simple* singular line. According as the two contact points in this case are distinct or coincident, the singular line is known as a *double tangent* or an *inflectional tangent.* The curve of Fig. 8,

$$(6) \qquad 4\,u_1^3 + 27\,u_2u_3^2 = 0 \qquad (y = x^3),$$

has the line $x_2 = 0$ (the axis of x) as an inflectional tangent, and the curve of Fig. 9,

$$(7) \qquad u_1^3 + u_1^2u_2 - u_2u_3^2 \quad 0,$$

has the same line as a double tangent.

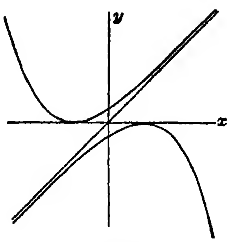


FIG. 9

Comparison of the simple singular lines with the simple singular points shows that a double point and a double tangent are dual and that a cusp and an inflectional tangent are dual.

At the cusp in Fig. 6, the point tracing the curve turns on itself and proceeds in the opposite direction. On the inflectional tangent in Fig. 8, the line enveloping the curve ceases to rotate in one direction and begins rotating in the opposite direction. As a matter of fact, the contact point on an inflectional tangent is a so-called *point of inflection*, and conversely.*

Pluecker's Equations. Let n and ν be the order and class, d and δ the number of double points and double tangents, and r and ρ the

* A complete discussion of this question would lead us too far afield. It suffices to note that equations (5) of § 3 show that the tangent at a point of inflection is a singular line, and the subsequent critique guarantees that the point of inflection is a contact point on this line.

number of cusps and inflectional tangents of an irreducible algebraic curve with only simple singular points and simple singular lines. Then two of the four famous equations of Pluecker read

$$\nu = n(n-1) - 2d - 3r, \tag{8a}$$

$$n = \nu(\nu - 1) - 2\delta - 3\rho. \tag{8b}$$

From these and the two remaining equations * may be deduced the equality of the two expressions,

$$\begin{aligned} p &= \tfrac{1}{2}(n-1)(n-2) - d - r, \\ p &= \tfrac{1}{2}(\nu-1)(\nu-2) - \delta - \rho. \end{aligned} \tag{9}$$

Since it can be shown further that the common value, p, can never be negative, it follows that the maximum values of the numbers, $d + r$ and $\delta + \rho$, of the singular points and singular lines are, respectively,

$$\tfrac{1}{2}(n-1)(n-2), \qquad \tfrac{1}{2}(\nu-1)(\nu-2). \tag{10}$$

Thus a curve of order (class) 3 can have at most one singular point (line), and a curve of order (class) 4 has at most three singular points (lines).

Since the integer p is the difference between the maximum possible number and the actual number of singular points, or singular lines, it is frequently called the *deficiency* of the curve. An equally common name for it is the *genus*. It is evident that a curve is of genus zero only when it has the maximum number of singular points and singular lines. A nondegenerate conic is, for example, a curve of genus zero.

Example 1. The curve of Fig. 6:

$$f \equiv x_1^3 - x_2^2 x_3 = 0. \tag{4}$$

Since $n = 3$, there is at most one singular point. This is readily found to be at the cusp $(0, 0, 1)$ with $x_2 = 0$ as tangent. Thus $n = 3$, $d = 0$, $r = 1$. Hence, by $(8a)$, $\nu = 3$. But then $(8b)$ becomes $2\delta + 3\rho = 3$, whence $\delta = 0$, $\rho = 1$. Our curve is therefore of class 3 and has one inflectional tangent. As a matter of fact, the equation of the curve in line coordinates is

$$\phi \equiv 4u_1^3 + 27u_2^2 u_3 = 0,$$

* Namely,

$$\begin{aligned} \rho &= 3n(n-2) - 6d - 8r, \\ r &= 3\nu(\nu-2) - 6\delta - 8\rho. \end{aligned}$$

and it is easily shown that $(0, 0, 1)$ is an inflectional tangent with contact point $u_2 = 0$.* The curve is of genus zero.

Example 2. The curve of Fig. 9:

$$\phi \equiv u_1^3 + u_1^2 u_2 - u_2 u_3^2 = 0. \tag{7}$$

This curve of the third class has, as its only singular line, the double tangent $(0, 1, 0)$ with the contact points $u_1 \pm u_3 = 0$. Since $\nu = 3$, $\delta = 1$, $\rho = 0$, it follows that the curve is of the fourth order. Hence we have, from $(8a)$, $2d + 3r = 9$. Inasmuch as $n = 4$, there can be at most three singular points and therefore the curve has three cusps and no double points. The genus is zero.

The equation of the curve in point coordinates is found to be

$$4x_1^4 - 4x_1^3 x_2 - (8x_1^2 - 36x_1x_2 + 27x_2^2)x_3^2 + 4x_3^4 = 0,$$

or

$$27y^2 + 4x(x^2 - 9)y - 4(x^2 - 1)^2 = 0.$$

Of the three cusps, one is (in the metric plane) at the point at infinity in the direction of the y-axis, with the line at infinity as cuspidal tangent. The other two are imaginary and are situated in the points $(9, 8, \pm 3\sqrt{3}\,i)$.

EXERCISES

1. Prove in detail the properties of the curve of Example 1 of the text.
2. The same for the curve of Example 2.
3. The curve of order three in the example in the text of § 5, namely

$$x_1^3 + x_2^3 + x_3^3 = 0,$$

has no singular points. Show that it has as singular lines the nine inflectional tangents with the contact points

$$\begin{array}{lll} (\ \ 1,\ -1,\ \ 0), & (\ \ \omega,\ -1,\ \ 0), & (\ \ \omega^2,\ -1,\ \ 0), \\ (-1,\ \ 0,\ \ 1), & (-1,\ \ 0,\ \ \omega), & (-1,\ \ 0,\ \ \omega^2), \\ (\ \ 0,\ \ 1,\ -1), & (\ \ 0,\ \ \omega,\ -1), & (\ \ 0,\ \ \omega^2,\ -1), \end{array}$$

where ω is an imaginary cube root of unity.

Prove that these nine points of inflection lie by threes on twelve lines, and that a line determined by two points of inflection always contains a third. The diagram for the evaluation of a three-rowed determinant will be found of value in picturing schematically the nine points and twelve lines.

4. Find the singular points and singular lines of the hypocycloid of four cusps:

$$u^2 + v^2 = a^2u^2v^2 \quad \text{or} \quad (x^2 + y^2 - a^2)^3 + 27a^2x^2y^2 = 0.$$

Ans. Six cusps: $(\pm a, 0)$, $(0, \pm a)$, and the circular points at infinity; four

* In the metric plane the line at infinity is the inflectional tangent and the contact point on it is in the direction of the y-axis.

double points: $(\pm ai, \pm ai)$, $(\pm ai, \mp ai)$; three double tangents: the two axes and the line at infinity.

5. Show that the curve

$$y^2 = x^3 - x^2$$

has an *isolated double point,* that is, a real double point at which the tangents are imaginary. Plot the curve in the metric plane.

6. Exhibit a curve with an isolated double tangent.

7. Unicursal Curves. The curve of the third order of Fig. 7,

$$y^2 = x^3 + x^2, \tag{1}$$

is cut by the line,

$$y = tx, \tag{2}$$

through its double point O in just one point P other than O, provided the line is not one of the two tangents at O. In these two cases, the point P comes into coincidence with O. Consequently, if we agree to think of the double point as two superimposed points, one corresponding to each of the two tangents, there is established a perfect one-to-one correspondence between the points of the curve and the lines of the pencil at O. The equations of the correspondence are obtained by solving equations (1) and (2) for the coordinates (x, y) of P in terms of the parameter t of the pencil. We thus get, as a parametric representation of the points of the curve,*

$$x = t^2 - 1, \qquad y = t^3 - t.$$

The corresponding parametric representation of the lines of the curve is readily found to be

$$u = -\frac{3t^2 - 1}{(t^2 - 1)^2}, \qquad v = \frac{2t}{(t^2 - 1)^2}.$$

There is, in this case, a perfect one-to-one correspondence between the points of the curve and the tangents to the curve.† Hence these equations establish a perfect one-to-one correspondence between the tangents and the values of the parameter t (including $t = \infty$).

Suppose now that the equations

$$\begin{aligned} x_1 &= \psi_1(t) = a_0t^n + a_1t^{n-1} + \cdots + a_n, \\ x_2 &= \psi_2(t) = b_0t^n + b_1t^{n-1} + \cdots + b_n, \\ x_3 &= \psi_3(t) = c_0t^n + c_1t^{n-1} + \cdots + c_n, \end{aligned} \tag{3}$$

* To obtain the point at infinity (0, 1, 0) on the curve, t must be allowed to become infinite.

† The curve has no double tangents, only inflectional tangents. If a double tangent had been present, we should have found it necessary to think of it as two superimposed tangents, one corresponding to each of the contact points on it.

where the a's, b's, c's are real constants and a_0, b_0, c_0 are not all zero, establish between the values of the complex parameter t (inclusive of $t = \infty$) and the points (x_1, x_2, x_3) of an irreducible algebraic curve a perfect one-to-one correspondence, provided only that each singular point is replaced by the proper number of superimposed points. There is, then, a similar one-to-one correspondence between the tangents to the curve and the values of t. The equations of this correspondence are the parametric equations for (u_1, u_2, u_3) corresponding to equations (3), namely,*

(4) $u_1 = \psi_2\psi_3' - \psi_3\psi_2', \qquad u_2 = \psi_3\psi_1' - \psi_1\psi_3', \qquad u_3 = \psi_1\psi_2' - \psi_2\psi_1'.$

An algebraic curve of this type is called *unicursal* or *rational.*

The order of the unicursal curve (3) is n. For, it meets an arbitrary line $(d|x) = 0$ in the n points which correspond to the n values of t which are the roots of the polynomial equation

$$d_1\psi_1(t) + d_2\psi_2(t) + d_3\psi_3(t) = 0.$$

Similar reasoning, applied to (4), would indicate that the class of the curve is $n + (n - 1)$ or $2n - 1$. This is, however, not the case, for the coefficients of the terms of degree $2n - 1$ in the polynomials in (4) all vanish. The coefficients of the terms of degree $2n - 2$ are found to be

$$b_1c_0 - b_0c_1, \qquad c_1a_0 - c_0a_1, \qquad a_1b_0 - a_0b_1.$$

These coefficients are not, in general, all zero and the class of the curve is, then,

$$\nu = 2(n - 1).$$

An algebraic curve has in general at most double points. Restricting ourselves to this case and substituting $n = n$, $\nu = 2(n - 1)$, $r = 0$, in

$$\nu = n(n - 1) - 2d - 3r,$$

we find that

$$d = \frac{(n - 1)(n - 2)}{2}.$$

Thus the curve has the maximum number of double points, and is therefore of genus zero.

This result is always valid. *Every unicursal curve is of genus zero.* Moreover it is true, conversely, that *every curve of genus zero is unicursal.*

The most important unicursal curves for our purposes are the non-degenerate conics.

* See last footnote on p. 210; also Ex. 5.

EXERCISES

1. Employing the method used to treat the cubic curve (1), establish parametric representations of the following nondegenerate conics:

(a) $y^2 = 2mx$; (c) $y^2 = x^2 - 3x + 2$;

(b) $x_1^2 + x_2^2 - x_3^2 = 0$; (d) $\Sigma a_{ij}x_ix_j = 0$.

2. Find the equations of the conics defined by the following parametric representations:

(a) $u_1 = 2t - 1, \quad u_2 = t^2 + 1, \quad u_3 = t^2 - 1$;

(b) $x_1 = -t^2 + t + 1, \quad x_2 = t^2 - t + 1, \quad x_3 = t^2 + t - 1$.

3. The equations $x_1 = t^4$, $x_2 = t^2$, $x_3 = 1$ evidently constitute a parametric representation of the conic $x_2^2 = x_1x_3$. But, for them, $n = 4$. Explain the paradox. Why was it not met with in the text?

4. Show that the curve of Fig. 6 is unicursal and establish for it a rational parametric representation.

5. Prove directly that the coordinates of the tangent to the curve (3) at the point x are given by (4).

Suggestion. Find the equation of the tangent as the limit of the secant.

CHAPTER XIV

PROJECTIVE, AFFINE, AND METRIC PROPERTIES OF CONICS

1. Projective Properties. Conjugate Points and Lines. An arbitrary nondegenerate conic is represented by the equations

(1) $\sum a_{ij}x_ix_j = 0, \qquad |a_{ij}| \neq 0; \qquad \sum b_{ij}u_iu_j = 0, \qquad |b_{ij}| \neq 0,$

where

$$b_{ij} = A_{ij} \qquad \text{or} \qquad a_{ij} = B_{ij},$$

according as the first or the second of the two equations is the one initially given.

Two points r and s are said to be conjugate to one another with respect to the nondegenerate conic (1) *if*

$$\sum a_{ij}r_is_j = 0 \qquad or \qquad \sum a_{ij}s_ir_j = 0. \tag{2 a}$$

It should be recalled that, since $a_{ij} = a_{ji}$, $\sum a_{ij}r_is_j = \sum a_{ij}s_ir_j$.

If one of the two points, say r, lies on the conic, the other, s, lies on the tangent at r; for the condition that s lie on the tangent at r is precisely (2 a).

Suppose that neither r nor s lies on the conic. The points of intersection of their line with the conic are

$$x' = r + \mu' s, \qquad x'' = r + \mu'' s, \tag{3}$$

where μ' and μ'' are the roots of the quadratic equation

$$\sum a_{ij}r_ir_j + 2\,\mu \sum a_{ij}r_is_j + \mu^2 \sum a_{ij}s_is_j = 0.$$

Recalling that the sum of the roots of the equation $a + b\,\mu + c\,\mu^2 = 0$ is zero when and only when $b = 0$, we conclude that r and s are conjugate if and only if

$$\mu' + \mu'' = 0,$$

and hence if and only if r and s are separated harmonically by the points (3).

THEOREM 1 a. *Two points, neither of which lies on the conic, are conjugate if and only if they separate harmonically the intersections of their line with the conic. Two points, the first of which is on the conic, are conjugate when and only when the second lies on the tangent at the first.*

Consider a *secant* of the conic, that is, *any line which is not a tangent.* The points of the secant are conjugate in pairs, inasmuch as they separate harmonically in pairs the points in which the secant intersects the conic. Hence:

THEOREM 2 *a*. *The pairs of conjugate points on a secant form an involution whose double points are the intersections of the secant with the conic.*

What can be said about the pairs of conjugate points on a tangent?

Conjugate Lines. The two lines r and s are conjugate with respect to the nondegenerate conic (1) if and only if

$$(2\,b) \qquad \sum b_{ij}r_is_j = 0 \qquad \text{or} \qquad \sum b_{ij}s_ir_j = 0.$$

The following Theorems, dual to Theorems 1 *a* and 2 *a*, we leave to the reader to demonstrate.

THEOREM 1 *b*. *Two lines, neither of which is a tangent to the conic, are conjugate if and only if they separate harmonically the tangents to the conic drawn from their point of intersection. Two lines, the first of which is a tangent, are conjugate when and only when the second passes through the point of contact of the first.*

THEOREM 2 *b*. *The pairs of conjugate lines through a point which is not on the conic form an involution whose double lines are the tangents drawn from the point.*

EXERCISES

1. Prove Theorem 1 *b*. **2.** Prove Theorem 2 *b*.

3. Show that a point is conjugate to itself with respect to the conic if and only if it lies on the conic. State the dual.

2. Poles and Polars. We continue our study of the projective properties of the nondegenerate conic

$$(1) \quad \sum a_{ij}x_ix_j = 0, \qquad |a_{ij}| \neq 0; \qquad \sum b_{ij}u_iu_j = 0, \qquad |b_{ij}| \neq 0,$$

where, as before, $b_{ij} = A_{ij}$ or $a_{ij} = B_{ij}$ according to which of the equations is the first one given.

The theory of poles and polars can be based either on the theory of conjugate points or on that of conjugate lines.

Based on Conjugate Points. Let a point r be given. An arbitrary point x is conjugate to r with respect to the conic (1) if and only if

$$(2\,a) \qquad \sum a_{ij}r_ix_i = 0.$$

Thus, the locus of the points conjugate to a given point P with respect to a nondegenerate conic is a line L. *The line L is called the polar of the point P with respect to the conic.*

THEOREM 1 *a*. *The polar of the point r with respect to the conic* (1) *is the line* (2 *a*).

If P is on the conic, the polar of P is the tangent at P. If P is not on the conic, two points which are conjugate to P and hence determine the polar of P are the points of contact of the tangents from P to the conic.

THEOREM 2 *a*. *The polar of a point not on the conic is the secant joining the points of contact of the tangents from the point. The polar of a point on the conic is the tangent at the point.*

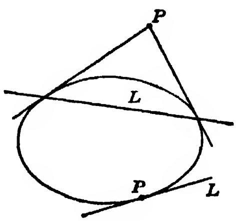


FIG. 1

If a line L is given, there is a unique point P which has L as its polar. For, if L is a tangent, P must be the point of contact of L, and if L is a secant, P must be the point of intersection of the tangents at the points in which L meets the conic.

The point P whose polar is a given line L is called the pole of L. Hence:

THEOREM 3 *a*. *The pole of a secant is the point of intersection of the tangents at the points in which the secant meets the conic. The pole of a tangent is the point of tangency.*

Based on Conjugate Lines. The line u is conjugate to the fixed line r if and only if

$$\sum b_{ij} r_i u_j = 0. \tag{2 b}$$

Hence the envelope of the lines conjugate to a given line L with respect to a nondegenerate conic is a point P. *The point P is called the pole of the line L with respect to the conic.*

THEOREM 1 *b*. *The pole of the line r with respect to the conic* (1) *is the point* (2 *b*).

If L is a tangent, the pole of L is the point of contact of L. If L is a secant, two lines which are conjugate to L and hence determine the pole of L are the tangents at the points in which L meets the conic. Hence:

THEOREM 2 *b*. *The same as Theorem* 3 *a*.

If a point P is given, there is a unique line L whose pole is P. The line L is defined as the *polar* of P.

We leave the proof to the reader. He will thereby establish

THEOREM 3 *b*. *The same as Theorem* 2 *a*.

It is clear that the two developments of poles and polars are, in effect, identical. As an immediate consequence of them, we have the theorem:

THEOREM 4. *A point lies on its polar if and only if it is on the conic. A line contains its pole if and only if it is tangent to the conic.*

Further Properties of Poles and Polars. Let P_1 and P_2 be two points. If P_1 lies on the polar of P_2, P_1 is conjugate to P_2. Hence P_2 is conjugate to P_1 and therefore lies on the polar of P_1.

THEOREM 5. *If P_1 lies on the polar of P_2, P_2 lies on the polar of P_1. If L_1 goes through the pole of L_2, L_2 goes through the pole of L_1.*

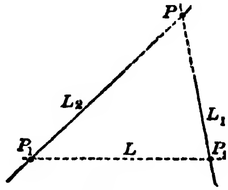


FIG. 2

Let L be the line joining P_1 and P_2, and let P be its pole. Since P_1 and P_2 lie on the polar L of P, P lies on the polars of P_1 and P_2 and hence is their point of intersection.

THEOREM 6. *The pole of the line joining two points is the point of intersection of the polars of the two points. The polar of the point of intersection of two lines is the line joining the poles of the two lines.*

The following theorems can now be readily deduced.

THEOREM 7. *If a number of points lie on a line, their polars all go through a point, the pole of the line. If a number of lines go through a point, their poles all lie on a line, the polar of the point.*

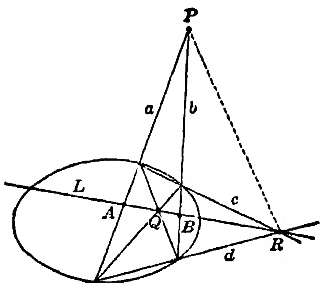


FIG.

THEOREM 8. *The polars of the points of a range are the lines of a pencil. The poles of the lines of a pencil are the points of a range.*

Constructions. To construct the polar of a point P, not on the conic, draw through P two lines a, b meeting the conic in four points. Join these points in the remaining possible ways, and denote by Q and R the new points of intersection of the lines drawn. The line QR is then the polar of P. For, since in the complete quadrilateral $a\ b\ c\ d$ the set of lines at R is harmonic, the sets of points on

a and b are harmonic; therefore the points A and B are conjugate to P and the line joining them is the polar of P.

Once the polar L of P has been found, the tangents from P can be constructed as the lines which join P to the points in which L intersects the conic. This is the simplest construction known for the tangents from an external point to a conic.

To construct the pole of a line L, think of L as determined by two points P_1, P_2 which lie on it but not on the conic. The pole of L will be the point of intersection of the polars of P_1 and P_2.

EXERCISES

1. Prove geometrically and analytically that two lines are conjugate if and only if each contains the pole of the other.

2. Prove the first half of Theorem 5 analytically and the second half geometrically.

3. Prove the second half of Theorem 6.

4. Establish Theorems 7 and 8.

5. Find the pole of the line $(2, -3, 1)$ with respect to the conic

$$3u_1^2 - 2u_2^2 + 4u_2u_3 - 2u_1u_3 + u_1u_2 = 0.$$

6. Find the polar of the point $(5, 1, 2)$ with respect to the conic of Ex. 5 without finding the equation of the conic in point coordinates

7. Carry through the construction for the pole of a secant.

8. Construct as simply as possible the tangent to a conic at a given point on the conic.

9. The same for the contact point of a conic on a given tangent.

10. Prove geometrically that, if a collineation carries a nondegenerate conic Q into a conic Q', it carries conjugate points (lines) with respect to Q into conjugate points (lines) with respect to Q', and a point and line which are pole and polar with respect to Q into a point and a line which are pole and polar with respect to Q'.

3. Involutory Correlations. The polar of the point x with respect to the nondegenerate conic

$$\sum a_{ij}x_ix_j = 0, \qquad a_{ij} = a_{ji}, \quad |a_{ij}| \neq 0, \tag{1}$$

has the equation

$$\sum_{j=1}^{3} a_{1j}x_j\, X_1 + \sum_{j=1}^{3} a_{2j}x_j\, X_2 + \sum_{j=1}^{3} a_{3j}x_j\, X_3 = 0,$$

and hence the coordinates

$$\rho u_1 = \sum_{j=1}^{3} a_{1j}x_j, \qquad \rho u_2 = \sum_{j=1}^{3} a_{2j}x_j, \qquad \rho u_3 = \sum_{j=1}^{3} a_{3j}x_j.$$

Thus, the transformation of a point x into its polar u with respect to the conic (1) is the *correlation*

$$(2) \qquad \rho u_i = \sum_{j=1}^{3} a_{ij}x_j, \qquad (i = 1, 2, 3), \qquad a_{ij} = a_{ji}, \qquad |a_{ij}| \neq 0.$$

Conversely, every correlation of the form (2), that is, every correlation for which $a_{ij} = a_{ji}$, is a transformation of pole into polar with respect to a nondegenerate conic (1).

The correlation (2) is involutory. For this is true of the equivalent transformation of pole into polar; the pole of the polar of a point is the point itself.

On the other hand, the general correlation

$$\rho u'_i = \sum_{j=1}^{3} a_{ij}x_j, \qquad (i = 1, 2, 3), \quad |a_{ij}| \neq 0$$

is involutory, that is, identical with its inverse,

$$\sigma u_i = \sum_{j=1}^{3} a_{ji}x'_j, \qquad (i = 1, 2, 3),$$

if and only if

$$a_{ij} = k\, a_{ji}, \qquad (i, j = 1, 2, 3).$$

But then

$$a_{ji} = k\, a_{ij}, \qquad (i, j = 1, 2, 3).$$

Hence

$$(k^2 - 1)a_{ij} = 0, \qquad (i, j = 1, 2, 3),$$

and

$$k^2 = 1.$$

If $k = -1$, we have $a_{ij} = -a_{ji}$ and hence, by Ch. I, § 6, Ex. 4, $|a_{ij}| = 0$, — a contradiction. There remains only the case $k = 1$, and this gives rise to the special correlations (2).

THEOREM 1. *The transformations of pole into polar with respect to nondegenerate conics are identical with the involutory correlations of the plane.*

EXERCISES

1. Using only the fact that the transformation of pole into polar is an involutory correlation, prove Theorems 5–8 of § 2.

2. Show that the polars of the vertices and the poles of the sides of a triangle are respectively the sides and vertices of a new triangle. The two triangles are known as *conjugate* or *polar* triangles with respect to the given conic.

4. Self-Conjugate Triangles. Projective Classification of Conics. A triangle is said to be *self-conjugate* with respect to a nondegenerate

conic if each vertex and the opposite side are pole and polar with respect to the conic.

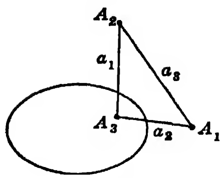


FIG. 4

If the polar of a vertex of a triangle is the opposite side, the vertex is conjugate to each of the other two vertices, and conversely. Dually, the pole of a side of a triangle is the opposite vertex if and only if the side is conjugate to each of the other two sides.

THEOREM 1. *A necessary and sufficient condition that a triangle be self-conjugate with respect to a given nondegenerate conic is that each two vertices be conjugate points with respect to the conic, or that each two sides be conjugate lines with respect to the conic.*

The theorem implies the existence of infinitely many triangles self-conjugate with respect to the given conic. As a first vertex, A_1, may be taken any point not on the conic, and as the other two vertices, A_2 and A_3, any pair of distinct conjugate points on the polar of A_1.

Reduction of the Equation of a Conic to Normal Form. Let the equation in point coordinates of a nondegenerate conic be

$$(1) \qquad \sum a_{ij}x_ix_j = 0, \qquad a_{ij} = a_{ji}, \qquad |a_{ij}| \neq 0.$$

Let $A_1A_2A_3$ be a real triangle self-conjugate with respect to the conic. Introduce new projective coordinates x'_1, x'_2, x'_3, referred to $A_1A_2A_3$ as the new triangle of reference. Inasmuch as the transformation from the old to the new coordinates is linear, the equation of the conic in the new coordinates is of the form

$$(2) \qquad \sum a'_{ij}x'_ix'_j = 0, \qquad a'_{ij} = a'_{ji}, \qquad |a'_{ij}| \neq 0.$$

Since the points A_1, A_2, A_3 are conjugate in pairs with respect to the conic, the new coordinates of each two of them satisfy the condition

$$\sum a'_{ij}r'_is'_j = 0.$$

It follows in the case of A_1: (1, 0, 0) and A_2: (0, 1, 0) that $a'_{12} = a'_{21} = 0$. Similarly, $a'_{23} = a'_{32} = 0$, and $a'_{31} = a'_{13} = 0$. Thus equation (2) becomes

$$(3) \qquad a'_{11}x'^2_1 + a'_{22}x'^2_2 + a'_{33}x'^2_3 = 0. \qquad a'_{11}a'_{22}a'_{33} \neq 0.$$

Two cases now arise. If a'_{11}, a'_{22}, a'_{33} are all of the same sign, they may be assumed all positive. Then a second change of coordinates, defined by the linear transformation,

$$(4\,a) \qquad \rho x''_1 = \sqrt{a'_{11}}\,x'_1, \qquad \rho x''_2 = \sqrt{a'_{22}}\,x'_2, \qquad \rho x''_3 = \sqrt{a'_{33}}\,x'_3,$$

reduces (3) to

$$(5\,a)\qquad x_1''^2 + x_2''^2 + x_3''^2 = 0.$$

If a'_{11}, a'_{22}, a'_{33} are not all of the same sign, we may assume, without loss of generality, that a'_{11} and a'_{22} are positive and a'_{33} negative.* The transformation of coordinates,

$$(4\,b)\quad \rho x_1'' = \sqrt{a'_{11}}\,x'_1, \qquad \rho x_2'' = \sqrt{a'_{22}}\,x'_2, \qquad \rho x_3'' = \sqrt{-a'_{33}}\,x'_3,$$

is then real and reduces (3) to

$$(5\,b)\qquad x_1''^2 + x_2''^2 - x_3''^2 = 0.$$

It is evident from (5 *b*) and (5 *a*) that the two cases arise according as the given conic has, or has not, real points.

THEOREM 2. *By a change of projective coordinates the equation in point coordinates of a nondegenerate conic is reducible to*

$$(6)\qquad x_1^2 + x_2^2 + x_3^2 = 0 \qquad \textit{or} \qquad x_1^2 + x_2^2 - x_3^2 = 0,$$

according as the conic has not, or has, a real trace.

COROLLARY. *The corresponding equations in line coordinates are*

$$(7)\qquad u_1^2 + u_2^2 + u_3^2 = 0, \qquad u_1^2 + u_2^2 - u_3^2 = 0.$$

The transformation of coordinates which reduces equation (1) directly to the normal form (5 *a*) or (5 *b*) is the product of the transformation from the coordinates x to the coordinates x' and the transformation (4 *a*) or (4 *b*). The first of these transformations introduces the chosen self-conjugate triangle as the triangle of reference; the second preserves this triangle of reference and associates with it a *particular* point as unit point; see Ch. X, § 2, Ex. 4.

Thus far we have thought of the linear transformation which carries the *equation* (1) of a conic without a real trace into the *equation* (5 *a*) as a change of coordinates. We can equally well think of it as a collineation which carries the *conic* (1) into the new *conic* (5 *a*). Hence, an arbitrarily chosen conic without a real trace can be carried by a collineation into the particular conic $(x|x) = 0$, and vice versa.

It follows that, if two conics without real traces are given, there exists a collineation which carries the first conic into $(x|x) = 0$, a second

* We assume two of the three coefficients in (3) positive. If the third, or negative, coefficient is not that of $x_3'^2$, it can be made to be by renaming the vertices of the self-conjugate triangle.

collineation which carries $(x|x) = 0$ into the second conic, and hence a collineation which carries the first conic into the second.

Since each of the two conics can be carried by a collineation into the other, each of them has the same projective properties as the other. Consequently, to ascertain the projective characteristics of all the conics without real traces, we need only to know those of one. In other words, *there is in projective geometry only one type of conic without a real trace.*

The foregoing arguments apply equally well to the conics with real traces. Hence:

THEOREM 3. *There are only two types of nondegenerate conics in the projective plane, those with real traces and those without real traces.*

We remind the reader that we are dealing with *real* conics with respect to the group of *real* collineations. The facts are still simpler in the case of complex conics, treated with respect to complex collineations; see Ex. 7.

The Concept of Equivalence. The foregoing discussion can be clarified by the introduction of the important geometric concept known as *equivalence.*

DEFINITION. *Two geometric configurations are said to be equivalent with respect to a certain group of transformations if there exists a transformation of the group which carries the one configuration into the other.*

Our results, stated in terms of this concept, are as follows.

THEOREM 4. *Each two real nondegenerate conics with real traces (or without real traces) are equivalent with respect to the group of real collineations.*

EXERCISES

1. Show that the triangle with vertices in the points (0, 1, 1), (1, 0, 1), (1, 1, 0) is self-conjugate with respect to the conic

$$x_1^2 + x_2^2 + x_3^2 + 2x_2x_3 + 2x_3x_1 - 6x_1x_2 = 0.$$

2. Reduce the equation of the conic in Ex. 1 to normal form by introducing the triangle as new triangle of reference. Leave the unit point the same.

3. Determine a triangle which is self-conjugate with respect to the conic

$$2u_1^2 - u_2^2 + u_3^2 - 2u_1u_2 - 4u_2u_3 = 0$$

and has the line $(1, 2, -1)$ as one of its sides.

4. Prove directly that the equation in line coordinates of a nondegenerate conic can be reduced to one of the forms (7).

5. Show that the equation in point coordinates of a nondegenerate conic

with a real trace can always be reduced, by a proper change of projective coordinates, to the form

$$x_2x_3 + x_3x_1 + x_1x_2 = 0.$$

Begin by introducing as the new triangle of reference a triangle which is inscribed in the conic.

6. State and prove the dual of the theorem of the preceding exercise.

7. Prove that each two complex nondegenerate conics are equivalent with respect to the group of complex collineations, that is, that there is only one type of nondegenerate conic in complex projective geometry.

5. Affine Properties of Conics. Affine properties of a conic deal with the relationship of the conic to the fixed line, or line at infinity, of affine geometry.

DEFINITION. *A nondegenerate conic which is tangent to the fixed line is a parabola. A nondegenerate conic which intersects the fixed line in two distinct points is a hyperbola or an ellipse according as the points are real or conjugate-imaginary.*

By a *center* of a nondegenerate conic is meant a point in which the conic is symmetric, that is, a point which is midway between the two intersections of every line through it with the conic. In other words, a center is a finite point which is conjugate to every point at infinity. But the only point with this property is the pole of the line at infinity, and it is finite only when the conic is not a parabola.

THEOREM 1. *A parabola has no center. An ellipse or a hyperbola has a unique center, the pole of the line at infinity.*

Accordingly, we call ellipses and hyperbolas *central conics.*

A tangent to a nondegenerate conic at a point on the fixed line, if it is not the fixed line itself, is an *asymptote.*

THEOREM 2. *A central conic has two distinct asymptotes. A parabola has no asymptotes.*

THEOREM 3. *The asymptotes of a central conic intersect in the center.*

The latter theorem follows from the fact that the asymptotes are the tangents at the points in which the fixed line meets the conic, and hence intersect in the pole of the fixed line.

Since an affine transformation carries the points at infinity of a central conic C into the points at infinity of the transformed conic C', it carries the asymptotes of C into the asymptotes of C', and the center of C into the center of C'. Thus the theories of asymptotes and centers really are affine.

By a *diameter* of a nondegenerate conic we shall mean a finite line through the pole of the fixed line.

Diameters of Central Conics. The diameters of a central conic are the lines through its center. By § 1, Th. 2 *b*, we have

THEOREM 4. *The diameters of a central conic are conjugate in pairs. The pairs of conjugate diameters form an involution whose double lines are the asymptotes.*

The theorem implies, in particular, that an asymptote is a self-conjugate diameter.

THEOREM 5. *The pole of a diameter is the point at infinity in the direction of the conjugate diameter. The polar of a point at infinity is the diameter which is conjugate to the diameter in the direction of the point.*

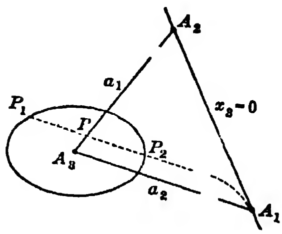


FIG. 5

The pole of a diameter is surely a point at infinity, since every diameter contains the pole of the line at infinity. Two lines conjugate to the diameter a_1 of Fig. 5, and therefore intersecting in the pole of a_1, are the diameter a_2 conjugate to a_1 and the line at infinity. Hence the pole, A_1, of a_1 is the point at infinity on a_2.

Inasmuch as A_3 is the pole of $x_3 = 0$, and A_1 and A_2 are respectively the poles of a_1 and a_2, the triangle $A_1A_2A_3$ is self-conjugate with respect to the conic.

THEOREM 6. *A pair of conjugate diameters and the fixed line constitute a self-conjugate triangle.*

Since A_1 is conjugate to all points on a_1, a finite point P on a_1 lies midway between the points P_1 and P_2 in which the line A_1P meets the conic. But P_1P_2 is the chord of the conic on A_1P, and A_1P is parallel to a_2. Hence:

THEOREM 7. *A diameter bisects the chords parallel to the conjugate diameter and is parallel to the tangents at the intersections of the conjugate diameter with the conic.*

The proof of the latter half of the theorem we leave to the reader.

Diameters of a Parabola. According to definition, a diameter of a parabola is a *finite* line through the point at infinity of the parabola.

There are no conjugate diameters, inasmuch as the line through the point at infinity of the parabola conjugate to a given diameter is always the line at infinity.

The following theorems are readily established.

THEOREM 8. *The pole of a diameter is the point at infinity in the direction of the tangent at the finite intersection of the diameter with the parabola.*

THEOREM 9. *A diameter bisects the chords parallel to the tangent at its finite intersection with the parabola.*

Affine Classification of Nondegenerate Conics. In the reduction of the equation

$$(1) \qquad \sum a_{ij}x_ix_j = 0, \qquad |a_{ij}| \neq 0,$$

to a simple form, we are restricted to affine transformations of coordinates, or what is the same thing, to triangles of reference which have the fixed line, $x_3 = 0$, as their third side.

If the conic (1) is a central conic, it is natural to introduce as the new triangle of reference a triangle consisting of two conjugate diameters and the fixed line. Since this triangle is self-conjugate with respect to the conic, the equation of the conic in the new coordinates is of the form

$$(2) \qquad a'_{11}x'^2_1 + a'_{22}x'^2_2 + a'_{33}x'^2_3 = 0, \qquad a'_{11}a'_{22}a'_{33} \neq 0.$$

Instead of (2) we can write, on the introduction of nonhomogeneous coordinates and a simplified notation for the coefficients,

$$(3) \qquad Ax'^2 + By'^2 = 1.$$

If A and B are both positive, equation (3) can be reduced by the affine transformation of coordinates,

$$x'' = \sqrt{A}\, x', \qquad y'' = \sqrt{B}\, y',$$

to

$$(4\,a) \qquad x''^2 + y''^2 = 1.$$

Similarly, if A and B are both negative, (3) can be reduced to

$$(4\,b) \qquad x''^2 + y''^2 = -1.$$

If A and B are opposite in sign, we can assume that A is positive and B negative and hence reduce (3) to

$$(4\,c) \qquad x''^2 - y''^2 = 1.$$

THEOREM 10. *By a change of affine coordinates the equation of a central conic can be reduced to*

$$x^2 + y^2 = 1 \quad or \quad x^2 + y^2 = -1 \quad or \quad x^2 - y^2 = 1,$$

according as the conic is an ellipse with a real trace or an ellipse without a real trace or a hyperbola. The corresponding equations in line coordinates are

$$u^2 + v^2 = 1, \qquad u^2 + v^2 = -1, \qquad u^2 - v^2 = 1.$$

If the conic (1) is a parabola, a self-conjugate triangle having the fixed line as a side does not exist. We take as the new triangle of reference a triangle $A_1A_2A_3$ (Fig. 6) consisting of a diameter, the tangent at the finite point of intersection of the diameter with the parabola, and the fixed line. Since $A_1 : (1, 0, 0)$ and $A_3 : (0, 0, 1)$ lie on the conic and are both conjugate to $A_2 : (0, 1, 0)$, the equation of the parabola in the new coordinates,

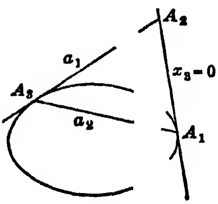


FIG. 6

$$\sum a'_{ij}x'_ix'_j = 0, \qquad |a'_{ij}| \neq 0,$$

becomes

$$a'_{22}x_2'^2 + 2\,a'_{13}x'_1x'_3 = 0, \qquad a'_{22}a'_{13} \neq 0,$$

or, in nonhomogeneous coordinates,

$$y'^2 = 2\,mx', \qquad m \neq 0.$$

The affine transformation of coordinates $x'' = m\,x'$, $y'' = y'$ reduces this equation to

$$(4\,d) \qquad y''^2 = 2\,x''.$$

THEOREM 11. *The equation of a parabola can be reduced by a proper change of affine coordinates to the form*

$$y^2 = 2\,x.$$

The corresponding equation in line coordinates is

$$v^2 = 2\,u.$$

Interpreting the affine transformations which we have employed as transformations of the plane, instead of as transformations of coordinates, we conclude from Theorems 10 and 11, by arguments analogous to those in the projective case, that there exist in the affine plane

only four types of nondegenerate conics: one type of hyperbola, one type of parabola, and two types of ellipses.

THEOREM 12. *Each two real ellipses with real traces (without real traces), or each two real hyperbolas, or each two real parabolas, are equivalent with respect to the group of real affine transformations.*

EXERCISES

1. Deduce a simple condition that the nondegenerate conic $\Sigma b_{ij}u_iu_j = 0$ be a parabola. The same for $\Sigma a_{ij}x_ix_j = 0$.

2. Find the asymptotes and center of the conic

$$x^2 + 3\,xy - 4\,y^2 + 2\,x - 10\,y = 0.$$

3. Show that the asymptotes of the central conic $\Sigma a_{ij}x_ix_j = 0$ are parallel to the lines represented by the equation

$$a_{11}x^2 + 2\,a_{12}xy + a_{22}y^2 = 0.$$

4. Find the diameter of the conic

$$x^2 - 3\,xy + 2\,y^2 = 4$$

which is conjugate to the axis of x.

5. By application of Theorem 4, show that the diameters of

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

which are of slopes λ, λ' are conjugate if and only if

$$\lambda\lambda' = \frac{b^2}{a^2}.$$

6. Prove the latter half of Theorem 7.

7. Establish Theorems 8 and 9.

8. Prove that the tangents to a central conic at the extremities of a chord meet on the diameter bisecting the chord. Is the proposition true for a parabola?

9. Show that a cross ratio of four diameters of a central conic is equal to the corresponding cross ratio of the four conjugate diameters.

10. Determine an affine transformation carrying the ellipse

$$2\,x^2 - 2\,xy + 5\,y^2 - 2\,x - 8\,y + 4 = 0$$

into

$$x^2 + y^2 - 1 = 0.$$

11. Find an affine change of coordinates which reduces the equation of the parabola

$$x^2 - 2\,xy + y^2 + 3\,x + 2\,y - 2 = 0$$

to the form

$$y^2 = 2\,x.$$

12. Show that each two complex central conics are equivalent with respect to the group of complex affine transformations. How many kinds of nondegenerate conics are there in complex affine geometry?

6. Metric Properties of Conics. The metric geometry of conics has to do with their relationships to the circular points at infinity, I and J.

Parabolas. By an *axis* of a conic is meant a line of symmetry, that is, a finite line which bisects the chords of the conic perpendicular to it. An axis of a parabola is, therefore, necessarily a diameter which is perpendicular to the tangent at the finite intersection of the diameter with the parabola. A diameter VP with this property must, with the corresponding tangent VQ, separate harmonically the isotropic lines through V (Fig. 7). In other words, the pole Q of the diameter must be the harmonic conjugate of P with respect to I and J. Hence Q, and therefore the diameter, is uniquely determined.

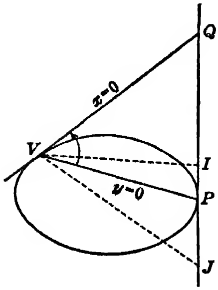


Fig. 7

Theorem 1. *A parabola has a single axis, the polar of the point which is the harmonic conjugate of the point at infinity on the parabola with respect to I and J.*

The finite point V in which the axis meets the parabola is the *vertex* of the parabola.

If rectangular Cartesian coordinates x, y, referred to the axis of an arbitrary parabola as x-axis and the tangent at the vertex as y-axis, are introduced, the equation of the parabola must, by reasoning similar to that of § 5, take the form

$$(1) \qquad y^2 = 2\,mx, \qquad m \neq 0.$$

We may assume that the axis of x has been so directed that m is positive. Then no two of the parabolas represented by (1) are metrically equivalent. It is not legitimate, for example, to claim the equivalence of the two parabolas

$$y^2 = 2\,m_1x, \qquad y'^2 = 2\,m_2x', \qquad m_1 \neq m_2,$$

by the transformation $m_2x' = m_1x$, $y' = y$, for this transformation is not a rigid motion.

Theorem 2. *There are ∞^1 nonequivalent types of real parabolas with respect to the group of real rigid motions.*

Central Conics. Inasmuch as we are dealing only with real conics and the points I and J are conjugate-imaginary, a conic which passes through one of these points must pass through the other.

A conic which contains I and J is a *circle*. The asymptotes of a circle are isotropic lines, and hence each two conjugate diameters of a circle are perpendicular.

Since an axis of a conic is a line which bisects the chords perpendicular to it, an axis of a central conic is a diameter which is perpendicular to the conjugate diameter. The latter diameter is then also an axis. Thus the axes of a central conic appear in pairs, as the pairs of mutually perpendicular conjugate diameters. Inasmuch as the conjugate diameters form an involution with the asymptotes as double lines, there is, except in the case of a circle, just one pair of mutually perpendicular conjugate diameters (Ch. IX, § 8, Th. 1), and they bisect the angles between the asymptotes.

THEOREM 3. *A central conic, not a circle, has two axes, the bisectors of the angles between the asymptotes. Every diameter of a circle is an axis.*

Since two perpendicular axes of a central conic and the line at infinity form a self-conjugate triangle, the equation of the conic referred to the axes, as axes of coordinates, assumes the form

$$Ax^2 + By^2 = 1, \qquad AB \neq 0. \tag{2}$$

THEOREM 4. *There are ∞^2 nonequivalent types of real central conics with respect to the group of real rigid motions.*

If A and B are of opposite signs, we assume $A > 0$ and $B < 0$ and set $A = 1/a^2$, $B = -1/b^2$, thus obtaining the hyperbolas

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1. \tag{2 a}$$

If A and B are of the same sign, we have the ellipses

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \pm 1, \tag{2 b}$$

with, or without, real traces. When $A = B$, we get the circles.

It is evident from our results that, whereas there are only a finite number of nonequivalent types of real conics in projective or affine geometry, there is an infinity of nonequivalent types of real conics in metric geometry, in fact, an infinity which depends on two parameters. The two parameters are reflected geometrically in the shape and size of the conic.

Foci and Directrices. The *foci* of a conic are the finite points of intersection of the isotropic tangents. The *directrices* are the polars of the foci.

Since a parabola is tangent to the fixed line, it has but two isotropic tangents and hence only one focus and one directrix. The isotropic tangents are conjugate-imaginary; the focus and the directrix are consequently real.

A circle has but two isotropic tangents, its asymptotes. Hence a circle has a single focus, its center, and a single directrix, the line at infinity.

A central conic, not a circle, has four distinct isotropic tangents, which are conjugate-imaginary in pairs. There are therefore four foci, two real and two conjugate-imaginary, and corresponding to them, two real and two conjugate-imaginary directrices.

THEOREM 5. *The foci of a central conic, not a circle, lie on the axes and are equidistant in pairs from the center.*

Let F and F' be the real foci, G and $\bar{G}$ the conjugate-imaginary foci. By a previous theorem (Ch. VIII, § 5, Ex. 4), FF' and $G\bar{G}$ are mutually perpendicular lines whose point of intersection C is equidistant from F and F' and from G and $\bar{G}$. To prove that C is the center and FF' and $G\bar{G}$ are the axes, it suffices to show that FF' and $G\bar{G}$ are conjugate diameters.

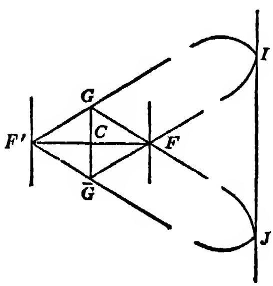


FIG. 8

The pole of FF' is the point of intersection of any two lines which are conjugate to FF'. The line through F conjugate to FF' is the harmonic conjugate of FF' with respect to the isotropic tangents FG and $F\bar{G}$ and is therefore perpendicular to FF'. Similarly, the line through F' conjugate to FF' is perpendicular to FF'. Since these two lines conjugate to FF' are parallel to $G\bar{G}$, the pole of FF' is the point at infinity in the direction of $G\bar{G}$. In the same way it may be shown that the pole of $G\bar{G}$ is the point at infinity in the direction of FF'. Consequently, FF' and $G\bar{G}$ are conjugate diameters.

COROLLARY. *The directrices corresponding to the foci lying on an axis are parallel to, and equidistant from, the other axis.*

The directrices which are the polars of F and F' go through the pole of FF' and so are parallel to $G\bar{G}$. That they are equidistant from $G\bar{G}$ follows from the fact that F and F', as well as the conic, are symmetric in $G\bar{G}$.

THEOREM 6. *The focus of a parabola lies on the axis, the directrix is perpendicular to the axis, and the vertex is equally distant from the focus and the directrix.*

The theorem can be established geometrically by methods analogous to those used in proving Theorem 5.

We could now proceed as in Ch. VIII, § 9, to obtain the metric properties which bear on foci and directrices.

EXERCISES

1. Find the axis and vertex of the parabola

$$x^2 + 4\,xy + 4\,y^2 - 2\,x = 0.$$

2. Find the slopes of the axes of the conic

$$2\,x^2 - 4\,xy - y^2 + 2\,x - 3\,y + 1 = 0,$$

by writing the equation of the involution of the pairs of conjugate diameters and determining from it the pair of perpendicular conjugate diameters.

3. Determine the foci of the conic

$$5\,x^2 - 4\,y^2 - 20\,x - 24\,y + 4 = 0.$$

4. Prove Theorem 6.

5. Find the focus of the parabola $uv + u + v = 0$, without finding the equation of the parabola in point coordinates.

6. Show that the line joining the pole of a focal chord of a conic to the focus is perpendicular to the focal chord.

7. The line joining a point to the center of a circle is perpendicular to the polar of the point with respect to the circle. Establish this statement and hence show that, if a triangle with finite vertices is self-conjugate with respect to a circle, the intersection of the altitudes must be the center of the circle.

7. Degenerate Conics. We agree to extend the definitions which we have introduced in this chapter for nondegenerate conics to degenerate conics, in so far as they are applicable.*

Projective Properties. We discuss, first, poles and polars and conjugate points with respect to a degenerate point conic

$$(1) \qquad \sum a_{ij}x_ix_j = 0, \qquad a_{ij} = a_{ji},$$

of rank two. If r is not the singular point, the equation

$$(2) \qquad \sum a_{ij}r_ix_j = 0$$

* Since degenerate point conics can be considered as conics only from the point of view of point geometry, we can expect to extend to them only that portion of our developments which pertains to point geometry. Similarly, we can hardly expect of degenerate line conics properties which belong essentially to point geometry. We cannot hope, for example, to get far with a theory of conjugate lines for a degenerate point conic.

always represents a straight line, the polar of r, and the points of this line are the points conjugate to r.

If r is on (1), the polar of r is the tangent at r and is therefore that one of the constituent lines of (1) on which r lies. If r is not on (1), the polar of r is the harmonic conjugate, with respect to the lines constituting the conic, of the line joining the singular point to r; for, if the constitutent lines of (1) are $(c|x) = 0$, $(d|x) = 0$, the line joining their point of intersection to r and the polar of r are respectively

$$(c|r)(d|x) - (d|r)(c|x) = 0, \qquad (c|r)(d|x) + (d|r)(c|x) = 0.$$

It follows that two points, neither of which is on (1), are conjugate if and only if the lines joining them to the singular point separate the constituent lines of (1) harmonically, and that two points, the first of which is on (1) but is not the singular point, are conjugate if and only if the second lies on the same constituent line as the first.*

If r is the singular point of (1), equation (2) is illusory: *the polar of the singular point is undefined.* On the other hand, the coordinates of every point s satisfy the equation $\sum a_{ij}r_is_j = 0$: *conjugate to the singular point is every point in the plane.*

Since the only polars of points are the lines through the singular point, the transformation of point into polar is not one-to-one and has no proper inverse transformation of polar into pole.†

Two lines which separate harmonically the constituent lines of (1) may justifiably be called conjugate lines, inasmuch as each contains points whose polars are the other. But these are the only pairs of lines which may properly be called conjugate.

By a self-conjugate triangle we shall understand, here, a triangle each two of whose vertices are conjugate points. One of the vertices is then the singular point and the lines joining the other two to the singular point separate the lines of the conic harmonically.

The facts concerning a degenerate point conic of rank one,

$$(c|x)^2 = 0,$$

can readily be deduced from the polar equation

$$(c|r)(c|x) = 0.$$

The points of the conic have no polars, and the polar of every point

* This is essentially Theorem 1 *a* of § 1, paraphrased for our degenerate conic.

† It is worth while in this connection to think of the transformation of point into polar as a singular involutory correlation (§ 3) and to recall the facts concerning singular collineations; see Ex. 11, End of Ch. VII.

not on the conic is the line of the conic. Conjugate to a point of the conic is every point in the plane, and conjugate to a point not on the conic is every point of the conic. The self-conjugate triangles are those with two vertices on the conic. Conjugate lines do not exist.

To reduce the equation of a degenerate point conic of rank two to normal form, we take as the sides $x_1' = 0$, $x_2' = 0$ of the new triangle of reference two lines separating harmonically the lines of the conic. The equation of the conic, referred to this self-conjugate triangle, becomes

$$a_{11}'x_1'^2 + a_{22}'x_2'^2 = 0, \qquad a_{11}'a_{22}' \neq 0,$$

and may, by suitable choice of the new unit point, be reduced to

$$x_1''^2 + x_2''^2 = 0 \qquad \text{or} \qquad x_1''^2 - x_2''^2 = 0,$$

according as the lines of the conic are conjugate-imaginary or real.

If the rank of the degenerate point conic is one, its equation is obviously reducible to

$$x_1'^2 = 0.$$

Here, too, the new triangle of reference is self-conjugate with respect to the conic.

We leave to the reader the corresponding developments for degenerate line conics.

Affine Properties. According as the two points in which the line at infinity meets a degenerate point conic * are real and distinct, real and coincident, or conjugate-imaginary, the conic is known as a degenerate hyperbola, degenerate parabola, or degenerate ellipse.

Applying the definitions introduced for nondegenerate conics, we find that a degenerate ellipse or hyperbola has two asymptotes, its constituent lines, and a unique center, its singular point.

A degenerate parabola is symmetric in every point of a line, that is, it has a line of centers. It also possesses asymptotes, its constituent lines, for these are its tangents at its point at infinity.

There is only one affine property which may be extended to degenerate line conics. The point midway between the constituent points of a degenerate line conic, provided these points are finite, may justifiably be called the center of the conic.

* We leave aside, as lacking in interest, the case in which the fixed line belongs to the conic.

Metric Properties. A degenerate central conic has two axes, the bisectors of the angles between its constituent lines.* A degenerate parabola has as axes the line of centers and every line perpendicular to the line of centers.

The definitions of the squares of the eccentricities of a nondegenerate hyperbola as functions of an angle α between the asymptotes, namely $\sec^2 \alpha/2$ and $\csc^2 \alpha/2$, are readily shown to hold for a nondegenerate ellipse, other than a circle. These definitions apply equally well to a degenerate central conic, not a circle. To a degenerate circle we give only the one eccentricity zero, and to a degenerate parabola, the single eccentricity unity.

EXERCISES

1. Discuss the projective properties of degenerate line conics.

2. Determine the nonequivalent types of degenerate point conics in affine geometry and exhibit normal forms of their equations.

3. Discuss the question of the foci of a degenerate line conic; of a degenerate point conic.

8. Invariants of Conics. Let the quadratic form

$$\sum_{ij} a_{ij}x_i x_j, \qquad a_{ij} = a_{ji}, \tag{1}$$

be carried by the collineation

$$x_i = \sum_j d_{ij}x'_j, \qquad (i = 1, 2, 3), \quad |d_{ij}| \neq 0 \tag{2}$$

into the quadratic form

$$\sum_{rs} a'_{rs}x'_r x'_s, \qquad a'_{rs} = a'_{sr}, \tag{3}$$

and denote by Δ the determinant $|d_{ij}|$ of the collineation, and by A and A' the discriminants $|a_{ij}|$ and $|a'_{rs}|$ of the given and transformed forms. Then

$$A' = \Delta^2 A. \tag{4}$$

THEOREM 1. *The discriminant of a quadratic form is a relative invariant, of weight two,† with respect to the group of collineations.*

In establishing (4) we need the values of the a''s in terms of the a's and d's. These are obtained by actually transforming (1) into (3)

* A degenerate circle is an exception. The constituent lines in this case are the isotropics through the center and every line through the center is an axis.

† In saying that the invariant is of weight two, we mean simply that the power to which Δ appears in (4) is two. An invariant of weight zero is an absolute invariant.

by (2). Setting

$$x_i = \sum_r d_{ir}x'_r, \qquad x_j = \sum_s d_{js}x'_s$$

from (2) into (1), we get

$$\sum_{ij} a_{ij}x_ix_j \equiv \sum_{ijrs} a_{ij}d_{ir}d_{js}x'_rx'_s \equiv \sum_{rs}(\sum_{ij} d_{ir}a_{ji}d_{js})x'_rx'_s,$$

or

(5) $$\sum_{ij} a_{ij}x_ix_j \equiv \sum_{rs} a'_{rs}x'_rx'_s,$$

where

(6) $$a'_{rs} = \sum_{ij} d_{ir}a_{ji}d_{js}.$$

This expression for a'_{rs} may be analyzed as follows:

$$a'_{rs} = \sum_i d_{ir}h_{is}, \qquad \text{where} \qquad h_{is} = \sum_j a_{ji}d_{js}.$$

The second of these sets of relations says that the determinant $H = |h_{is}|$ is the product of the determinants A and Δ: $H = A\,\Delta$; the first says that A' is the product of Δ and H : $A' = \Delta\, H$. Hence $A' = \Delta\, A\, \Delta = \Delta^2 A$.

Our proof is, however, not quite complete. It remains to show that a'_{rs} and a'_{sr} are actually equal.* This follows readily from (6) by virtue of the fact that $a_{ij} = a_{ji}$.

Theorem 2. *The expression*

$$\sum_{ij} a_{ij}y_iz_j$$

is an absolute invariant of the quadratic form (1) *and the two points* y *and* z *with respect to the group of collineations.*

Since

$$y_i = \sum_r d_{ir}y'_r, \qquad z_j = \sum_s d_{js}z'_s.$$

we have

$$\sum_{ij} a_{ij}y_iz_j \equiv \sum_{ijrs} a_{ij}d_{ir}d_{js}y'_rz'_s \equiv \sum_{rs} (\sum_{ij} d_{ir}a_{ij}d_{js})y'_rz'_s,$$

and hence, by (6),

(7) $$\sum a_{ij}y_iz_j \equiv \sum a'_{ij}y'_iz'_j.$$

Geometric Interpretations. Our theorems furnish analytic proofs of certain fundamental propositions in the projective geometry of conics. The first theorem assures us that, if a collineation carries a point conic

* The definition of the discriminant of a quadratic form assumes that the form has been written as $\Sigma a_{ij}x_ix_j$, *where* $a_{ij} = a_{ji}$.

Q into a point conic Q', Q and Q' are both degenerate or both nondegenerate.

The second theorem guarantees (*a*) that two points conjugate with respect to Q are carried into two points conjugate with respect to Q'; (*b*) that the polar of a point with respect to Q is carried into the polar of the transformed point with respect to Q'; and (*c*) that a tangent to Q at a given point goes into a tangent to Q' at the transformed point.

It is obvious geometrically that a necessary condition that two point conics be projectively equivalent is that they have the same rank. Hence, *the rank of the matrix of a quadratic form is an invariant with respect to the group of collineations.* This invariant differs, evidently, in type from the invariants previously considered. It is known as an *arithmetic invariant.*

EXERCISES

1. Show that $B^2 - 4\,AC$ is a relative invariant of weight two of the conic *

$$(8) \qquad Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0,$$

with respect to the group of affine transformations. Give a geometrical reason why the weight must be an even number.

2. Prove that $A + C$ and $B^2 - 4\,AC$ are absolute invariants of the conic (8) with respect to the group of rigid motions. What is the geometrical significance of the vanishing of $A + C$?

3. Show that the roots of the equation

$$(B^2 - 4\,AC)\,z^2 - 4\,[(A + C)^2 + B^2 - 4\,AC]\,(z - 1) = 0$$

are the squares of the eccentricities of the conic (8).

EXERCISES ON CHAPTER XIV

1. Show that, if the vertices of a complete quadrangle lie on a nondegenerate conic, the diagonal triangle is self-conjugate with respect to the conic. State the dual.

2. Prove that every triangle self-conjugate with respect to a nondegenerate conic may be considered as the diagonal triangle of a complete quadrangle whose vertices are on the conic. State and prove the dual.

3. Let Q be a nondegenerate conic, and let L be a line not tangent to it. If $A_1, B_1, C_1, \ldots$ are points on L and $A_2, B_2, C_2, \ldots$ are the intersections with L of the polars of $A_1, B_1, C_1, \ldots$ with respect to Q, show that the pairs of points $A_1, A_2, B_1, B_2, C_1, C_2, \ldots$ form an involution.

4. Let a nondegenerate conic and two lines L and L' which are not conjugate with respect to the conic be given. Show that there is a unique point P' on L' which is conjugate to an arbitrarily chosen point P on L, and vice versa. Prove

* It is assumed that A, B, C are not all zero.

that the one-to-one correspondence thus established between the points of L and L' is projective. When will the point common to L and L' correspond to itself?

5. Let P be a point not on a nondegenerate conic, L its polar, and Q an arbitrary point on the conic. Show that the points in which L is met by PQ and the tangent at Q are conjugate points.

6. Show that the equation in line coordinates of an arbitrary parabola tangent to the coordinate axes is of the form

$$a_1uv + a_2u + a_3v = 0, \qquad a_1a_2a_3 \neq 0.$$

7. Show that the two chords connecting a point on a central conic to the ends of a diameter are parallel to two conjugate diameters.

8. Show that the sides of a parallelogram inscribed in a central conic are parallel to a pair of conjugate diameters.

9. Under what conditions will two central conics have the same pairs of conjugate diameters?

10. Prove that the pairs of points in which a diameter, not an asymptote, cuts two conjugate hyperbolas form a harmonic set.

11. Show that the isotropic lines through the center of a conic are conjugate diameters if and only if the conic is a rectangular hyperbola.

12. Where must a point be located in relation to a nondegenerate conic in order that each two perpendicular lines through it be conjugate with respect to the conic?

13. There is in general a unique line which is perpendicular to a given finite line and also conjugate to it with respect to a given nondegenerate conic. What are the exceptions?

14. Prove that, if each of two pairs of opposite vertices of a complete quadrilateral consists of conjugate points with respect to a nondegenerate conic, so also does the third pair.

15. Show that two triangles which are polars of each other (§ 3, Ex. 2) are in the relationship of Desargues. What special theorem results when one of the triangles is circumscribed about the given conic?

16. An *imaginary* nondegenerate parabola which is tangent to the line at infinity at J: $(1, i, 0)$ has no focus and no directrix. Its equation may be reduced to the form

$$(x + i\,y)^2 + a\,(x - i\,y) = 0, \qquad a \neq 0,$$

by introducing the contact point M of the isotropic tangent from I as origin, and the constant a may be made to take on the value unity by proper choice of rectangular axes at M. The parabola admits of a group of three rigid motions into itself, namely the group of rotations about M through the angles 0°, 120°, and 240°! Consequently, it is possible to inscribe in it infinitely many equilateral triangles!! What is the eccentricity of the parabola, as defined by § 8, Ex. 3?

17. An *imaginary* nondegenerate central conic which passes through J, but not through I, has no axes and only two foci and two directrices. The

directrices are isotropics, and the line joining the foci is an isotropic asymptote. The equation of the conic, referred to the center as origin and to the polar of I as the axis of x, is

$$x^2 - 2\,i\,xy + 3\,y^2 = 4\,k^2, \qquad k \neq 0.$$

The two foci F and F' then have the coordinates $(k, k\,i)$ and $(-k. -k\,i)$ and for every point P on the conic

$$\pm\, F\,P \pm F'\,P = 2\sqrt{2}\,k.$$

On the other hand, the conic is *not* the locus of a point moving so that the ratio of its distances from a focus and a directrix is constant, and both its eccentricities, as given by § 8, Ex. 3, are zero!

CHAPTER XV

PROJECTIVE THEORY OF CONICS

1. Projective Generation of Conics. First Method. Let it be required to find the locus of a point P which moves so that the cross ratio (L_1L_2, L_3L_4) of the four lines joining P to four given points, no three collinear, is constant and equal to k:

$$(1) \qquad (L_1L_2, L_3L_4) = k.$$

Take the four given points as the basic points A_1, A_2, A_3, D of a projective coordinate system and denote the coordinates of P by (x_1, x_2, x_3). The equations of L_1 and L_2 are, then,

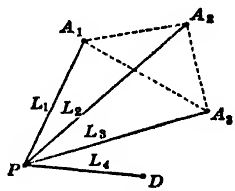


FIG. 1

$$L_1: \qquad \alpha \equiv x_3X_2 - x_2X_3 = 0,$$

$$L_2: \qquad \beta \equiv x_1X_3 - x_3X_1 = 0,$$

and the equations of L_3 and L_4 are the linear combinations of these equations which are satisfied respectively by (0, 0, 1) and (1, 1, 1), namely

$$L_3: \qquad x_1\,\alpha + x_2\,\beta = 0,$$

$$L_4: \qquad (x_1 - x_3)\,\alpha + (x_2 - x_3)\,\beta = 0.$$

Hence (1) becomes

$$\frac{x_2(x_1 - x_3)}{x_1(x_2 - x_3)} = k,$$

or

$$(2) \qquad x_2x_3 - k\,x_3x_1 + (k - 1)\,x_1x_2 = 0.$$

This equation represents a point conic which passes through each of the given points.* The conic is nondegenerate provided we assume that $k \neq 0, 1$.

THEOREM 1 *a*. *The locus of a point which moves so that a chosen cross ratio of the lines joining it to four fixed points, no three collinear, is constant, $\neq 0, 1$, is a nondegenerate point conic passing through the four points.*

* Strictly speaking, the given points do not belong to the locus, since the cross ratio in (1) is undefined when P coincides with one of them.

*Conversely, every nondegenerate point conic * can be generated in this manner.*

To prove the converse, we choose on the given conic four distinct points and take them as the basic points of our coordinate system (Fig. 2). The equation of the conic then is

$$a_1x_2x_3 + a_2x_3x_1 + a_3x_1x_2 = 0, \qquad a_1a_2a_3 \neq 0. \tag{3}$$

where

$$a_1 + a_2 + a_3 = 0.$$

Without loss of generality, we may set

$$a_1 = 1, \qquad a_2 = -k, \qquad a_3 = k - 1, \qquad k \neq 0, 1.$$

Equation (3) then becomes identical with (2) and the conic is the locus of a point P moving so that $(L_1L_2, L_3L_4) = k.$

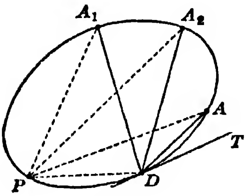


FIG. 2

If we allow P, moving on the conic, to approach a given point, say D, as its limit (Fig. 2), the cross ratio retains always the value k, while the secant PD approaches as its limit the tangent DT at D. Therefore

$$(DA_1\ DA_2,\ DA_3\ DT) = k.$$

We now restate the converse in the form in which it was originally discovered.

STEINER'S† THEOREM. *If four points on a nondegenerate conic are given, a chosen cross ratio of the four lines joining them to an arbitrary fifth point on the conic has a constant value. The corresponding cross ratio of the lines joining three of the given points to the fourth and the tangent at the fourth has the same value.*

We leave to the reader the proofs of the dual theorems.

THEOREM 1 *b*. *The envelope of a line which moves so that a chosen cross ratio of the points in which it intersects four given lines, no three concurrent, is constant, $\neq 0, 1$, is a nondegenerate line conic tangent to the four lines.*

THEOREM 2 *b*. *If four tangents to a nondegenerate conic are given, a chosen cross ratio of the four points in which they are met by an arbitrary fifth tangent has a constant value. The corresponding cross ratio of the*

* We exclude in this chapter conics without real traces.

† Steiner (1796–1863) was particularly interested in the projective generation of algebraic curves and surfaces.

points in which three of the given tangents intersect the fourth and the point of contact of the fourth has the same value.

EXERCISES

1. Prove Theorem 1 *b*.

2. Prove Theorem 2 *b*.

3. Prove geometrically that, if the cross ratio k in the locus problem is 0, 1, or ∞, the locus consists of a pair of opposite sides of the complete quadrangle determined by the four given points.

4. Show that Steiner's Theorem is true for a degenerate point conic, provided only that the cross ratio in question is defined.

2. Conics Determined by Points and Tangents. In the envelope problem of § 1, the constant value of the cross ratio can be prescribed as the cross ratio (P_1P_2, P_3P_4) of the points in which a fifth given line L_5 intersects the four given lines L_1, L_2, L_3, L_4 (Fig. 3). Then L_5 will be one of the positions of the moving line and the envelope will be tangent to it as well as to the four given lines. Hence there is a conic Q tangent to the five lines L_1, L_2, L_3, L_4, L_5.

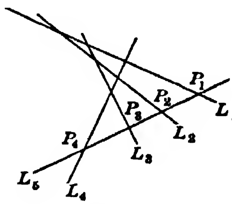


FIG. 3

Suppose now that a conic Q', tangent to the five lines, is given. Since it is assumed that no three of the lines are concurrent, Q' is nondegenerate. Hence Q' is the envelope of a line L moving so that the cross ratio of the points in which it intersects L_1, L_2, L_3, L_4 is a constant; since one position of L is L_5, the value of this constant is (P_1P_2, P_3P_4). Thus Q' answers to the same description as Q and therefore coincides with Q.

THEOREM 1 *a*. *Tangent to five lines, no three concurrent, there is just one line conic and it is nondegenerate.*

THEOREM 1 *b*. *Through five points, no three collinear, there passes just one point conic and it is nondegenerate.*

The constant value of the cross ratio in the envelope problem of § 1 can also be prescribed as the cross ratio (P_1P_2, P_3P_4), where P_1, P_2, P_3 are the points in which L_1, L_2, L_3 are met by L_4, and P_4 is a fourth point on L_4. The resulting conic will be tangent to L_1, L_2, L_3, L_4, and the point of tangency on L_4 will be P_4. We are thus led to the following theorem. The detailed proof is similar to that of Theorem 1 *a*.

THEOREM 2 *a*. *Tangent to three lines and to a fourth at a given point there is a unique line conic and it is nondegenerate. It is assumed that*

no three of the four lines are concurrent and that the point on the fourth does not lie on any of the other three.

THEOREM 2 *b.* *Passing through three points and tangent at a fourth to a given line there is a unique point conic and it is nondegenerate. It is assumed that no three of the four points are collinear and that the line through the fourth does not contain any of the other three.*

EXERCISES

1. Prove Theorem 1 *b*.

2. Prove Theorem 2 *b*.

3. Show that through five points, no *four* collinear, there passes just one point conic, nondegenerate or degenerate. How many conics are there through the five points, if just four are collinear? If all five are collinear?

4. Under what conditions will there be a *unique* point conic, nondegenerate or degenerate, passing through three points and tangent at a fourth to a given line?

3. One-Dimensional Perspective Correspondences. If a projective correspondence is established between two ranges of points lying on distinct lines, the point common to the two ranges will not, in general, correspond to itself. When it does, the ranges are said to be *perspective.* Similarly, two pencils of lines which are in projective correspondence so that the common line is self-corresponding are called *perspective.*

If two ranges of points are the projections of one another, their common point is self-corresponding and they are perspective. Conversely, if two ranges are perspective, the lines joining corresponding points are concurrent. For, if we think of the perspective correspondence as determined by three pairs of corresponding points $A_1 \leftrightarrow A_2$, $B_1 \leftrightarrow B_2$, $C_1 \leftrightarrow C_2$, where $A_1(A_2)$ is the point common to the two ranges, the projection from the point O in which the lines B_1B_2 and C_1C_2 intersect also orders A_1, B_1, C_1 to A_2, B_2, C_2 respectively, and hence establishes between the two ranges the given perspective correspondence.

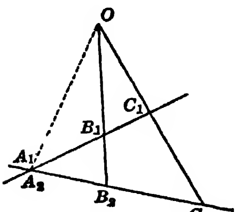


FIG. 4

THEOREM 1 *a.* *Two projective ranges of points are perspective if and only if the lines joining corresponding points are concurrent.*

The point of concurrency is known as the *center of perspective.*

THEOREM 1 *b.* *Two projective pencils of lines are perspective if and only if the points of intersection of corresponding lines are collinear.*

The line of collinearity is called the *axis of perspective.*

A range of points and a pencil of lines in projective correspondence are said to be perspective if each point of the range lies on the corresponding line of the pencil. The range of points A_1, B_1, C_1, . . . and the pencil of lines OA_1, OB_1, OC_1, . . . in Fig. 4 are perspective.

EXERCISES

1. Prove Theorem 1 *b*.

2. Show that, if three of the lines joining corresponding points of two projective ranges are concurrent, the ranges are perspective. State the dual.

4. Projective Generation of Conics. Second Method. We have just seen that the locus of the points of intersection of corresponding lines of two perspective pencils is a straight line, the axis of perspective. Strictly speaking, however, we should include also the line common to the two pencils, for it corresponds to itself and has in common with itself every one of its points. The locus then consists of two straight lines and is a degenerate point conic.

In the case of two pencils which are projective but not perspective, the locus is a nondegenerate point conic.

THEOREM 1 *a*. *The locus of the points of intersection of corresponding lines of two projective, nonperspective pencils of lines is a nondegenerate point conic.*

THEOREM 1 *b*. *The envelope of the lines joining corresponding points of two projective, nonperspective ranges of points is a nondegenerate line conic.*

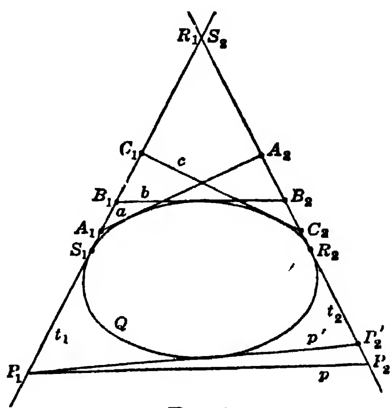


FIG. 5

We choose to prove the second of the two theorems. Let three pairs of corresponding points in the given correspondence between the ranges on the lines t_1 and t_2 be $A_1 \leftrightarrow A_2$, $B_1 \leftrightarrow B_2$, $C_1 \leftrightarrow C_2$, and let the lines joining these pairs of points be a, b, c. Since the ranges are not perspective, a, b, c are not concurrent (§ 3, Ex. 2). As a matter of fact, it is not difficult to show that no three of the five lines a, b, c, t_1, t_2 are

concurrent. Hence there is a unique nondegenerate conic Q tangent to them.

The line p joining two arbitrarily chosen corresponding points P_1 and P_2 of the two ranges will be tangent to Q, if it can be shown to be identical with the second tangent p' drawn from P_1 to Q (Fig. 5). Inasmuch as the two ranges are projective,

$$(1) \qquad (A_2B_2,\ C_2P_2) = (A_1B_1,\ C_1P_1).$$

Since the four tangents a, b, c, p' cut the two tangents t_1, t_2 in two sets of points whose corresponding cross ratios are equal (§ 1, Th. 2 b),

$$(2) \qquad (A_2B_2,\ C_2P_2') = (A_1B_1,\ C_1P_1).$$

From (1) and (2) we have

$$(A_2B_2,\ C_2P_2) = (A_2B_2,\ C_2P_2').$$

Hence P_2 coincides with P_2', and p is identical with the tangent p'.

Conversely, if p is a tangent to Q, relation (1) holds, by § 1, Th. 2 b, and P_1 and P_2 are a pair of corresponding points of the two ranges.

As a moves, always tangent to Q, toward t_1 as a limit, A_1 approaches the contact point S_1 of t_1 and A_2 approaches the common point of the ranges, considered as a point S_2 of the range on t_2. Thus, to the common point, considered as a point of the one range, corresponds the point of contact with Q of the line of the other range.

COROLLARY TO THEOREM 1 b. *The conic is tangent to the lines of the two ranges at the points which correspond to the point common to the two ranges.*

Let the reader now prove Theorem 1 a and derive the corollary to it.

COROLLARY TO THEOREM 1 a. *The conic goes through the vertices of the two pencils and is tangent at them to the lines which correspond to the line common to the two pencils.*

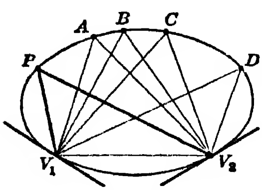


FIG. 6

To prove that every nondegenerate conic Q can be generated in the way described in Theorem 1 a, join two fixed points V_1 and V_2 on Q to an arbitrary point P on Q. As P traces Q, there is established between the pencils of lines at V_1 and V_2 a correspondence which becomes one-to-one without exception when the tangents at V_1 and V_2 are ordered to the common line of the two pencils. By Steiner's Theorem,

this correspondence preserves cross ratio and is therefore projective. Thus Q is the locus of the points of intersection of the corresponding lines of two projective pencils.

We may interpret this result as follows.

THEOREM 2 a. *The points of a nondegenerate conic subtend projective pencils of lines at two fixed points of the conic.*

THEOREM 2 b. *The tangents to a nondegenerate conic cut two fixed tangents in projective ranges of points.*

We are now in a position to complete the set of theorems begun in § 2.

THEOREM 3 a. *Tangent at two given points to given lines and passing through a third point there is a unique point conic and it is nondegenerate. It is assumed that the three given points are not collinear, and that each of the two given lines contains just one of the given points.*

Let V_1, V_2, C be the given points and a_1 and b_2 the given lines passing through V_1 and V_2 (Fig. 7). The pairs of lines $a_1 \leftrightarrow a_2$, $b_1 \leftrightarrow b_2$, $c_1 \leftrightarrow c_2$ establish between the pencils at V_1 and V_2 a projective, nonperspective correspondence. The locus of the point P in which an arbitrary pair of corresponding lines of the pencils intersect is a nondegenerate conic Q which goes through C, since C is one position of P, and is tangent at V_1 to a_1 and at V_2 to b_2.

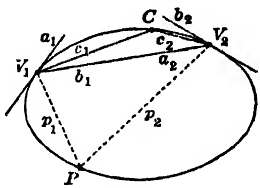


FIG. 7

Suppose now that Q' is a conic which satisfies the given requirements. Q' cannot be degenerate. (Why?) Hence Q' is the locus of points of intersection of corresponding lines of two projective pencils at V_1 and V_2. This projective correspondence is identical with that which gave rise to Q, since it has in common with it the three pairs of corresponding lines $a_1 \leftrightarrow a_2$, $b_1 \leftrightarrow b_2$, $c_1 \leftrightarrow c_2$. Therefore Q' is the same as Q.

THEOREM 3 b. *Dual of Theorem* 3 a.

EXERCISES

1. Establish Theorem 1 a and the corollary thereto.

2. Prove Theorem 2 b. **3.** State and prove Theorem 3 b.

4. *Maclaurin's Method of Generating a Conic.* A variable triangle moves so that its three sides turn about three given noncollinear points A, B, C and two of the vertices trace fixed lines a, b, not containing any of the given points. Show that, if A, B, and the point O of intersection of a and b are not collinear,

the locus of the third vertex is a nondegenerate conic which passes through the points A, B, A', B', O (Fig. 8).

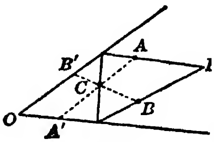


FIG. 8

5. State and prove the dual proposition.

6. Discuss the locus of P in Ex. 4 when A and B are collinear with O.

7. Mark on a straight line a series of equally spaced points, numbering the points consecutively, and repeat the construction on a second line. What can you say of the lines which join like-numbered points?

8. *Newton's Method of Generating a Conic.* Two constant angles move about fixed vertices so that a side of one always intersects a side of the other on a fixed line. Show that the point of intersection of the other two sides traces a point conic. When is the conic degenerate?

9. Describe the envelope of the lines joining corresponding points of the projective ranges on the lines L and L' of Ex. 4, End of Ch. XIV. Discuss all cases.

5. Continuation. Analytic Treatment. Let there be given two projective, nonperspective pencils of lines with vertices V and V' and let three pairs of corresponding lines have the coordinates a, b, c and a', b', c' so chosen that

$$c = a + b, \qquad c' = a' + b'.$$

Then

$$p = \mu\, a + b, \qquad p' = \mu\, a' + b'$$

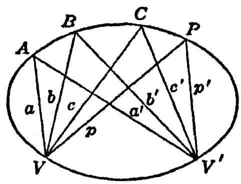


FIG. 9

are coordinates of an arbitrary pair of corresponding lines.

Since the point $P : (x_1, x_2, x_3)$ whose locus is desired lies on each of the lines p and p', we have

$$(1) \qquad \begin{aligned} (a|x)\,\mu + (b|x) &= 0, \\ (a'|x)\,\mu + (b'|x) &= 0. \end{aligned}$$

Eliminating μ, we find that P lies on the conic

$$(2) \qquad (a|x)(b'|x) - (a'|x)(b|x) = 0.$$

This conic goes through V since, for V, $(a|x)$ and $(b|x)$ are both zero. Similarly, the conic goes through V'.

To prove that the conic is tangent at V and V' to the lines corresponding to VV', we may assume, without loss of generality, that the three pairs of corresponding lines determining the projective corre-

spondence are as shown in Fig. 10. Then, since $a' = b$,* the equation of the conic becomes

$$(a|x)(b'|x) - (b|x)^2 = 0, \tag{3}$$

and the tangent to the conic at the point r is

$$(a|r)(b'|x) + (b'|r)(a|x) - 2(b|r)(b|x) = 0.$$

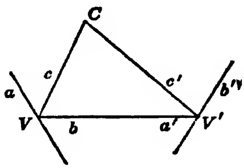


FIG. 10

When the r's are thought of as the coordinates of V, this equation reduces to $(a|x) = 0$. Hence the line a is the tangent at V, as was to be proved.

Any five points, no three collinear, may be made to play the rôles of V, V', A, B, C in Fig. 9. If, then, the coordinates a, b, c and a', b', c' are computed subject to the restrictions $c = a + b$, $c' = a' + b'$, equation (2) will represent the conic through the five points.

Figure 10 and equation (3) may be similarly applied to find the equation of the conic tangent at two given points to given lines and passing through a third point.

EXERCISES

1. Describe a method of finding the equation of a conic tangent at a given point to a given line and passing through three other points.

Find the equations of the following conics.

2. The conic passing through the five points $(-2, 0, 1)$, $(2, 0, 1)$, $(-1, 2, 1)$, $(1, 2, 1)$, $(0, 3, 1)$.

3. The conic tangent to the five lines $(1, 0, 1)$, $(-1, 0, 1)$, $(0, 1, 1)$, $(1, 0, 1)$, $(1, 1, 1)$.

4. The conic tangent at the point $(-1, -1)$ to $x + y + 2 = 0$ and passing through the points $(1, 0)$, $(1, 1)$, $(-1, 1)$.

5. The conic tangent to the lines $(1, 1, 2)$, $(3, -1, -3)$ at the points $(-1, -1, 1)$, $(1, 0, 1)$ and tangent to the line $(0, 3, 1)$.

6. The Theorems of Pascal and Brianchon. A conic is determined by five points or by five tangents. Under what conditions will six points lie on a conic or six lines be tangent to a conic? These and other important questions are answered by Pascal's and Brianchon's Theorems.

The theorems have to do with hexagons. By a hexagon is meant a figure consisting of six distinct points connected in a specific order by

* We can take $a' = b$ and still choose coordinates so that $c = a + b$ and $c' = a' + b'$.

six distinct lines, or a figure consisting of six distinct lines brought to intersection in a definite order in six distinct points. Figure 11 shows a hexagon determined by the six ordered vertices I, II, III, IV, V, VI, or by the six ordered sides 1, 2, 3, 4, 5, 6. A pair of vertices, or sides, which are separated by two vertices, or sides, are called opposite; for example, I and IV are opposite vertices and 3 and 6 are opposite sides.

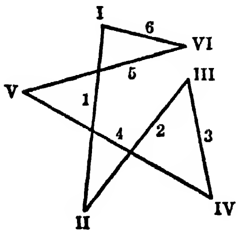


Fig. 11

Pascal's Theorem. *If a hexagon is inscribed in a nondegenerate conic, the points of intersection of the pairs of opposite sides are collinear. Conversely, if the pairs of opposite sides of a hexagon, determined by six points no three of which are collinear, intersect in points of a line, the six vertices of the hexagon lie on a nondegenerate conic.*

We assume a hexagon inscribed in a nondegenerate conic, as shown in Fig. 12, and prove that the points L, M, N in which the pairs of opposite sides intersect are collinear. By Steiner's Theorem, a cross ratio of the lines joining the points A, B, C, E to D is equal to the corresponding cross ratio of the lines joining the same points to F:

(1) $(DA\ DB, DC\ DE) = (FA\ FB, FC\ FE)$.

The first set of lines intersects the side 1 in the points A, B, C_1, L; the second set intersects the side 2 in A_2, B, C, M. Hence

(2) $\quad (A\ B, C_1\ L) = (A_2\ B, C\ M)$.

There is therefore a projective correspondence between the ranges of points on the lines 1 and 2, in which to A, B, C_1, L correspond respectively A_2, B, C, M. Since the point B common to the two ranges is self-corresponding, the ranges are perspective. Consequently, the lines AA_2, C_1C, and LM are concurrent. But AA_2 and C_1C intersect in N and so N lies on LM.

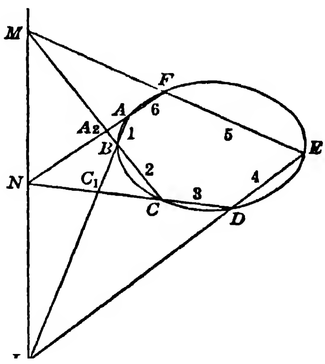


Fig. 12

The converse can be established by retracing steps. Since L, M, N are collinear, the lines AA_2, C_1C, LM are concurrent and the projective

correspondence established between the points of the lines 1 and 2 which is determined by $A \leftrightarrow A_2$, $C_1 \leftrightarrow C$, $L \leftrightarrow M$ is perspective: $B \leftrightarrow B$. Thus the cross ratios in (2) are equal and hence the cross ratios in (1) are also equal. Consequently, the conic which is the locus of a point P moving so that $(PA\ PB,\ PC\ PE) = k$, where k is the common value of the cross ratios in (1), passes through all six vertices of the hexagon.*

A hexagon whose pairs of opposite sides intersect in points on a line is called a *Pascal hexagon*, and the line, a *Pascal line*.

Six points, no three collinear, can be arranged in many different orders and so determine many different hexagons—in fact, sixty (Ex. 4). If one of these hexagons is a Pascal hexagon, the six points lie on a conic and hence all the hexagons are Pascal hexagons.

COROLLARY. *Six points, no three collinear, lie on a conic if and only if a hexagon having them as vertices is a Pascal hexagon.*

Let the student establish the dual results:

BRIANCHON'S THEOREM. *If a hexagon is circumscribed about a non-degenerate conic, the lines joining the pairs of opposite vertices are concurrent. Conversely, if the lines joining the opposite vertices of a hexagon, determined by six lines no three of which are concurrent, go through a point, the six sides of the hexagon are tangent to a nondegenerate conic.*†

A hexagon which has the property that the lines joining the pairs of opposite vertices go through a point is called a *Brianchon hexagon*, and the point, a *Brianchon point*.

COROLLARY. *A necessary and sufficient condition that six lines, no three concurrent, be tangent to a conic is that a hexagon having them as sides be a Brianchon hexagon.*

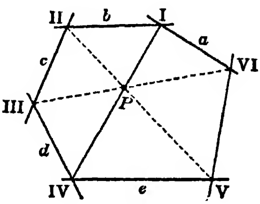


FIG. 13.

Construction Problems. Let it be required to construct additional lines tangent to the conic determined by five given lines.

Arrange the five given lines in a definite order, a, b, c, d, e, and bring a and b, b and c, c and d, d and e to intersection in the points I, II, III, IV (Fig. 13). Draw the line I IV and

* Which steps of the reasoning depend on the hypothesis that no three of the six vertices are collinear?

† Pascal discovered his famous theorem in 1640, when he was only 16 years old. It was 166 years later that Brianchon found the dual theorem!

choose on it a point P. Denote the points in which the lines II P and III P intersect e and a respectively by V and VI. The line V VI is then tangent to the conic determined by the five given lines.

By moving P along the line I IV a large number of tangents may be constructed. These will outline, however, only a portion of the conic. To obtain a finished picture, arrange the five given lines in other orders and repeat the construction.

The dual problem of constructing additional points on a conic determined by five points admits of an equally simple solution by means of Pascal's Theorem.

EXERCISES

1. Construct a sufficient number of tangents to a conic determined by five tangents to outline the conic completely.

2. Describe in detail a method of constructing additional points on a conic determined by five points.

3. Prove Brianchon's Theorem.

4. Show that six points, no three collinear, are the vertices of sixty distinct hexagons.*

5. Establish Maclaurin's method for generating a conic by proving, by Pascal's Theorem, that the points O, A', B', A, B, P of Fig. 8 lie on a conic.

6. Apply the dual method to Ex. 5 of § 4.

7. Continuation. Special Cases. If in Fig. 12 we allow the vertex F, moving on the conic, to approach the vertex A as a limit, the side 6 approaches the tangent at A and the hexagon is replaced in the limit by the inscribed pentagon $ABCDE$ (Fig. 14). On the other hand, since F stays on the conic, the points L, M, N remain always collinear and hence the points which they approach as limits are collinear.

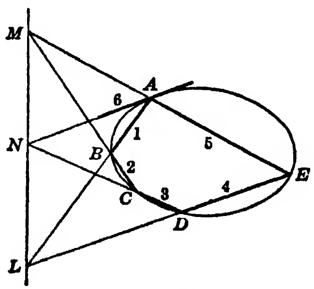


FIG. 14

THEOREM 1. *If a pentagon is inscribed in a nondegenerate conic, the point in which the tangent at one vertex intersects the opposite side and the points of intersection of the pairs of opposite remaining sides are collinear. Conversely, if a line through*

* If the six points lie on a conic, the sixty hexagons are all Pascal hexagons. The sixty Pascal lines pass by threes through twenty points; the twenty points lie by fours on fifteen lines, three of the lines going through each point.

a vertex of a pentagon, no three of whose vertices are collinear, meets the opposite side in a point collinear with the intersections of the opposite remaining sides, the line is tangent to the conic which circumscribes the pentagon.

By the hypothesis of the converse, the given line through the vertex A meets the side 3 on the line LM. But the tangent at A to the conic which circumscribes the pentagon also meets 3 on LM, by the direct theorem. Hence the given line through A is the tangent at A, and the converse is established.

The theorem furnishes a condition necessary and sufficient that a line through one of five given points, no three of which are collinear, be tangent to the conic passing through the five points. It also yields a simple construction for the tangent at one of the given points.

In Fig. 14 we can let approach one another the two vertices C and D opposite to the vertex A; or two vertices adjacent to A, say D and E. We thus obtain the theorems illustrated by Figs. 15 and 16, and corresponding converse theorems.

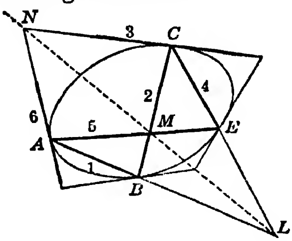


FIG. 15

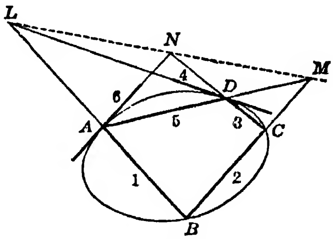


FIG. 16

In Fig. 15, not only will the tangents at A and C meet in a point on the line LM, but also the tangents at B and E. We thus obtain a theorem concerning two quadrilaterals.

THEOREM 2. *If a quadrilateral is inscribed in a conic and a second quadrilateral is circumscribed about the conic so that the ordered vertices of the first are respectively the points of contact of the ordered sides of the second, the points of intersection of the pairs of opposite sides of the two quadrilaterals are collinear.*

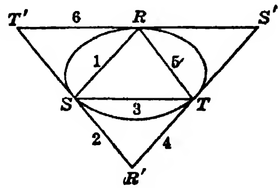


FIG. 17

If in Fig. 16 we let C, moving on the conic, approach B as its limit,

we obtain two triangles, one inscribed in the conic and the second consisting of the tangents to the conic at the vertices of the first (Fig. 17). Since the sides RS and $R'S'$, ST and $S'T'$, TR and $T'R'$ of the two triangles intersect in collinear points, the triangles are in the relationship of Desargues. We are thus led to the theorem:

THEOREM 3. *If RST and $R'S'T'$ are two triangles so situated that each vertex of the first lies on the opposite side of the second, there exists a conic circumscribing the first and inscribed in the second if and only if the two triangles are in the relationship of Desargues.*

EXERCISES

1. Construct the tangents to a conic determined by five points at each of the given points.

2. State and prove a condition necessary and sufficient that a point on one of five lines, no three of which are concurrent, be a contact point of the conic tangent to the five lines. Describe and illustrate a method for the construction of the contact point.

3. Discuss the limiting cases of Brianchon's Theorem which have to do with quadrilaterals.

4. Derive the dual of Theorem 3.

5. A nondegenerate conic is determined by four points and a tangent at one of them. (*a*) State and prove a condition necessary and sufficient that a fifth point lie on the conic and give a method for constructing additional points on the conic. (*b*) Derive a necessary and sufficient condition that a line through one of the remaining points be tangent to the conic.

6. A nondegenerate conic is fixed by three tangents and the points of contact on two of them. Find a necessary and sufficient condition (*a*) that a point on the remaining tangent be the contact point on the tangent; (*b*) that a fourth line be tangent to the conic.

8. Further Developments Concerning Projective Correspondences. Let there be given, on the lines L_1 and L_2, two perspective ranges of points, with center of perspective O and common point R. Let two pairs of corresponding points $A_1 \leftrightarrow A_2$, $B_1 \leftrightarrow B_2$ be joined crosswise, A_1 to B_2 and B_1 to A_2, and let X be the point of intersection of the joining lines. The line RX is clearly the harmonic conjugate, L, of the line RO with respect to L_1 and L_2 and is therefore independent of the particular pairs of corresponding points chosen. Hence, *all the points of intersection of lines joining crosswise two pairs of corresponding points lie on a line.*

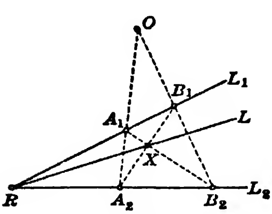


FIG. 18

In a correspondence between the two ranges which is projective but not perspective, the points $A, B, C, \ldots$ of intersection of the lines joining one pair of corresponding points $O_1 \leftrightarrow O_2$ crosswise with each of the remaining pairs of corresponding points $A_1 \leftrightarrow A_2$, $B_1 \leftrightarrow B_2$, $C_1 \leftrightarrow C_2, \ldots$ lie on a line L (Fig. 19). For, the points $A, B, C, \ldots$

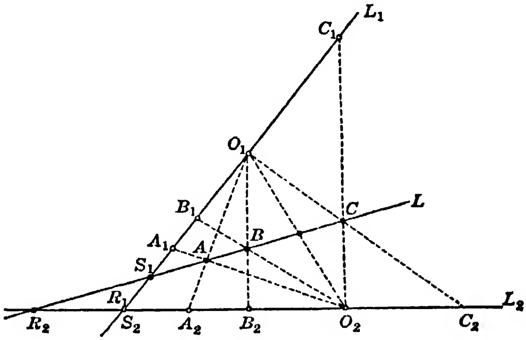


FIG. 19

are the intersections of the corresponding lines of the two pencils, O_1O_2, O_1A_2, O_1B_2, $O_1C_2, \ldots$ and O_2O_1, O_2A_1, O_2B_1, $O_2C_1, \ldots$, and these pencils are perspective; they are projective since the ranges on L_1 and L_2 are projective, and their common line O_1O_2 corresponds to itself.

To be able to conclude that *all* the points of intersection of lines joining crosswise two pairs of corresponding points of the two ranges lie on L, we must show that L is independent of the particular pair of points chosen as $O_1 \leftrightarrow O_2$. As a matter of fact, L is the line joining the points R_2 and S_1 which correspond to the point $R_1(S_2)$ common to the two ranges. The lines O_1R_2 and O_2R_1 are corresponding lines of the two pencils at O_1 and O_2, and hence their intersection, R_2, lies on L. Likewise S_1, as the intersection of O_1S_2 and O_2S_1, lies on L. Hence L is the line R_2S_1.

THEOREM 1. *The points of intersection of lines joining crosswise two pairs of corresponding points of two projective ranges lie on a line. If the ranges are not perspective, the line is the join of the points corresponding to the point common to the two ranges. If the ranges are perspective, it is the harmonic conjugate, with respect to the lines of the ranges, of the line joining the common point to the center of perspective.*

In both cases the line is called the *axis of the projectivity.*

Converse of Theorem 1. *If two ranges of points are in one-to-one correspondence so that the points of intersection of lines joining crosswise two pairs of corresponding points are collinear, the ranges are projective.*

In giving the proof we make use of Fig. 19, thinking of $O_1 \leftrightarrow O_2$, $A_1 \leftrightarrow A_2$, $B_1 \leftrightarrow B_2$, $C_1 \leftrightarrow C_2$, . . . as pairs of points in the given correspondence. Since A, B, C, . . . are, by hypothesis, collinear, the pencils of lines at O_1 and O_2 are perspective. Hence the two ranges are projective.

The duals of Theorem 1 and its converse we state as a single theorem.

Theorem 2. *Two pencils of lines which are in one-to-one correspondence are projective if and only if the lines joining points of intersection of two pairs of corresponding lines taken crosswise go through a point.*

This point is known as the *center of the projectivity.* Where is it located, when the pencils are not perspective? When the pencils are perspective?

The range of points A_1, B_1, C_1, . . . on L_1 in Fig. 19 is perspective with the range A, B, C, . . . on L, and this range is in turn perspective with the range A_2, B_2, C_2, . . . on L_2. Hence the projective correspondence between the ranges on L_1 and L_2 is equivalent to the succession of two perspective correspondences. In other words, two projective ranges can be connected by a succession of (two) perspectivities.

It can be shown, dually, that two projective pencils can be connected by a succession of (two) perspectivities.

If a pencil and a range which are projective are given, a range perspective with the given pencil (§ 3) can be connected as in Fig. 19 with the given range.

Theorem 3. *Two projective one-dimensional fundamental forms can be connected by a succession of perspectivities.*

EXERCISES

1. A projective correspondence between the ranges of points on two distinct lines is given by the prescription of three pairs of corresponding points. Show how to construct the point on the one line which corresponds to a given point on the other.

2. Prove Theorem 2 and answer the questions in the paragraph following it.

3. In a perspective correspondence between two pencils of lines, determined

by two pairs of distinct corresponding lines, the axis of perspective is inaccessible. How may it, then, be possible to construct the line of the one pencil which corresponds to a given line of the other?

4. By Th. 1 show that Pascal's Theorem remains true for a hexagon inscribed in a degenerate point conic consisting of two distinct straight lines, when each two successive vertices of the hexagon lie one on each line.

9. Ranges of Points and Pencils of Lines of the Second Order. A cross ratio of the lines joining four given points on a nondegenerate conic to a fifth point on the conic is independent of the position of the fifth point and is therefore, when the conic is given, a property of the four points alone. What is more natural than to call it *a cross ratio of the four points?*

What is the dual definition of the cross ratio of four tangents to a conic?

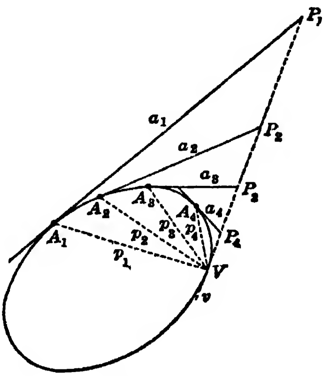


Fig. 20

Theorem 1. *A cross ratio of four points on a nondegenerate conic is equal to the corresponding cross ratio of the four tangents at the points.*

Let A_1, A_2, A_3, A_4 be the four points and a_1, a_2, a_3, a_4 the tangents at them. To prove that $(A_1A_2,\ A_3A_4) = (a_1a_2,\ a_3a_4)$, it is sufficient to prove (Fig. 20) that $(p_1p_2,\ p_3p_4) = (P_1P_2,\ P_3P_4)$. But this is evident inasmuch as p_i is the polar of P_i, $i = 1, 2, 3, 4$.

Projective Correspondences between Two Conics. The totality of points on a nondegenerate conic is known as *a range of points on a conic* or as *a range of the second order.* The totality of tangents to a nondegenerate conic is called *a pencil of tangents to a conic* or *a pencil of the second order.* The two together are known as *one-dimensional fundamental forms of the second order.*

Definition. *Two fundamental forms of the second order are in projective correspondence if their elements are in one-to-one correspondence so that corresponding cross ratios are equal.*

A simple case, suggested by Th. 1, is that in which the two forms consist respectively of the points and the tangents of the same conic, and to each point is ordered the tangent at the point.

THEOREM 2. *There is one and only one projective correspondence between two fundamental forms of the second order which orders to three given distinct elements of the one form three prescribed distinct elements of the second form.*

We consider the case of two ranges of points on two nondegenerate conics Q_1 and Q_2. Let A_1, B_1, C_1 be the given points of the range on Q_1, and let A_2, B_2, C_2 be the corresponding prescribed points of the range on Q_2. Join A_1, B_1, C_1 by the lines a_1, b_1, c_1 to a point V_1 on Q_1, and A_2, B_2, C_2 by the lines a_2, b_2, c_2 to a point V_2 on Q_2. There is a unique projective correspondence between the pencils of lines at V_1 and V_2 in which to a_1, b_1, c_1 correspond a_2, b_2, c_2. Hence there is one and only one projective correspondence between the ranges of points on Q_1 and Q_2 in which to A_1, B_1, C_1 correspond A_2, B_2, C_2.

Projective Fundamental Forms of the Second Order on the Same Conic. If, in the preceding proof, the conics Q_1 and Q_2 are one and the same conic Q, a simple method presents itself for the construction of the point P_2 of the second range which corresponds, in the projective correspondence determined by $A_1 \leftrightarrow A_2$, $B_1 \leftrightarrow B_2$, $C_1 \leftrightarrow C_2$, to a given point P_1 of the first range. Join A_1, B_1, C_1 to A_2, and A_2, B_2, C_2 to A_1, as shown in Fig. 21.* The projective correspondence between the two pencils of lines at A_2 and A_1 which is determined by $a_1 \leftrightarrow a_2$, $b_1 \leftrightarrow b_2$, $c_1 \leftrightarrow c_2$ is perspective, since the line A_1A_2 corresponds to itself. Thus the points of intersection of pairs of corresponding lines of the two pencils lie on a line, the line BC, or L, of the figure. Hence, if P_1 is a given point of the first range and P is the point in which A_2P_1 meets L, the point P_2 of the second range which corresponds to P_1 is the second point in which A_1P meets Q.

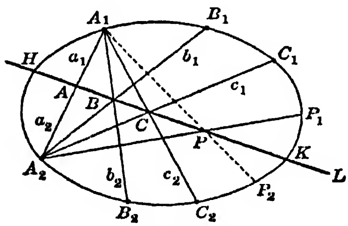


FIG. 21

It is evident from the construction that the points H and K in which L intersects Q are self-corresponding points and are the only self-corresponding points.

THEOREM 3. *A projective correspondence, other than the identity, between two ranges of points on the same conic has two self-corresponding, or double, points.*

* It is assumed that A_1 and A_2 are distinct. If A_1, B_1, C_1 coincide respectively with A_2, B_2, C_2, the given correspondence is the identity.

According as the two double points are coincident or distinct, that is, according as L is a tangent or a secant of Q, the projective correspondence is called *parabolic* or *nonparabolic*. Nonparabolic correspondences are further classified as *hyperbolic* or *elliptic* according as their double points are real or imaginary.

The line L is known as the *axis of projectivity*. It is determined by the double points. In particular, if the double points coincide, it is the tangent to Q at their point of coincidence.

The points $B, C, P, \ldots$ of the axis L are the points of intersection of the lines which join the pair of corresponding points $A_1 \leftrightarrow A_2$ crosswise with each of the remaining pairs of corresponding points $B_1 \leftrightarrow B_2$, $C_1 \leftrightarrow C_2$, $P_1 \leftrightarrow P_2, \ldots$. But the axis L is independent of the particular pair of corresponding points chosen as $A_1 \leftrightarrow A_2$. Hence:

THEOREM 4. *The points of intersection of lines joining crosswise two pairs of corresponding points of two projective ranges on the same nondegenerate conic lie on the axis of projectivity. Conversely, if two ranges of points on the same nondegenerate conic are in one-to-one correspondence so that the points of intersection of lines joining crosswise two pairs of corresponding points are collinear, the ranges are projective.*

The proof of the converse is similar to that of the converse of Theorem 1 of § 8.

EXERCISES

1. If the points of two projective ranges of points on the same nondegenerate conic Q are joined to a point V on Q, two projective pencils of lines are obtained with V as common vertex. Establish the following relationships between the projectivity between the pencils at V and the projectivity between the ranges on Q.

A. The double lines of the projectivity at V meet Q in the double points of the projectivity on Q.

B. The projectivity on Q is elliptic, parabolic, hyperbolic, or the identity, according as the projectivity at V is elliptic, parabolic, hyperbolic, or the identity.

C. The projectivity on Q is involutory if and only if the projectivity at V is involutory.

D. If the projectivity on Q is nonparabolic, a pair of corresponding points separate the double points in a constant cross ratio. This cross ratio is known as the *invariant* of the projectivity on Q and has the same value as the invariant of the projectivity at V.

2. Give a construction for a pair of corresponding lines in a projective correspondence between two pencils of tangents to the same conic, which is determined by three pairs of corresponding tangents. Hence prove the duals of Theorems 3 and 4.

10. Involutions on a Conic.

THEOREM 1. *Two pairs of distinct points on a nondegenerate conic form a harmonic set if and only if the lines determined by them are conjugate with respect to the conic.*

Let P_1, P_2 and Q_1, Q_2 be the given pairs of points and let p and q be the lines determined by them (Fig. 22). If p and q are conjugate with respect to the conic, q passes through the pole P of p and the points P, R and Q_1, Q_2 form a harmonic set. The lines P_1P, P_1R and P_1Q_1, P_1Q_2 then form a harmonic set and hence so do the points in which they meet the conic. But these are the given points P_1, P_2 and Q_1, Q_2.

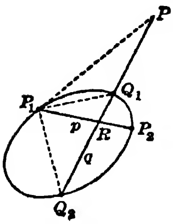


FIG. 22

Conversely, if P_1, P_2 and Q_1, Q_2 form a harmonic set and P is the point of intersection of the tangent at P_1 with q, the points P, R and Q_1, Q_2 form a harmonic set. Consequently, P is conjugate to R and, since it is also conjugate to P_1, it is the pole of p. Hence q is conjugate to p.

THEOREM 2. *A projective correspondence between two ranges of points on the same nondegenerate conic is involutory if and only if it is nonparabolic with invariant* -1.

This theorem follows from the analogous theorem for one-dimensional fundamental forms of the first order (Ch. IX, § 7) by virtue of the results of Ex. 1 of the preceding paragraph.

An involution in the points of a conic consists, then, of the pairs of points on the conic which separate two fixed points H, K harmonically. The two fixed points may be real or conjugate-imaginary, that is, the involution may be *hyperbolic* or *elliptic*.

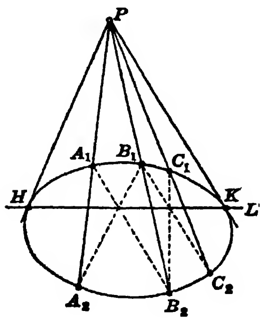


FIG. 23 *a*

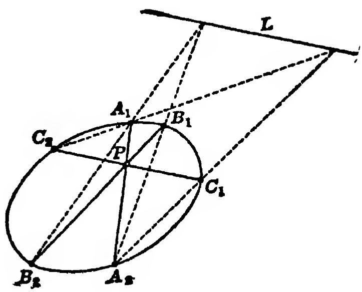


FIG. 23 *b*

The line joining the double points H, K is the axis L of the involution. Since the pairs of points in the involution all separate harmonically the double points, the lines determined by them are, by Th. 1, all conjugate to the axis and hence all go through the pole P of the axis. Conversely, if the lines joining the pairs of corresponding points of two projective ranges on a conic go through a point P, the points of each pair of points on the conic collinear with P are interchanged in pairs and the projective correspondence is, by definition, involutory.

THEOREM 3. *A necessary and sufficient condition that a projective correspondence between two ranges of points on a conic be involutory is that the lines joining corresponding points go through a point.*

This point is known as the *center* of the involution. The involution is hyperbolic if the center is *outside* the conic, that is, if the tangents from the center are real (Fig. 23 a). If the center is *inside* the conic, that is, if the tangents from it are imaginary, the involution is elliptic (Fig. 23 b).

Theorem 3 can be restated in a form which brings out clearly the inherent simplicity of an involution on a conic.

THEOREM 4. *A pencil of lines whose vertex is not on a nondegenerate conic cuts the conic in pairs of points in an involution. Every involution in the points of the conic can be thought of as generated in this way.*

We conclude also:

THEOREM 5. *An involution in the points of a nondegenerate conic is uniquely determined by its center or by its axis.*

The center of an involution can be thought of as the point of intersection of the lines joining the points of two pairs in the involution. Hence:

THEOREM 6. *An involution in the points of a nondegenerate conic is uniquely determined by two pairs of distinct points.*

In other words:

THEOREM 7. *There is a unique pair of points H, K on a nondegenerate conic which separate harmonically each of two given pairs of distinct points A_1, A_2 and B_1, B_2 on the conic.*

The points H, K are the intersections with the conic of the polar of the point of intersection, P, of the lines A_1A_2, B_1B_2. They are real or conjugate-imaginary according as P is outside or inside the conic.

But P is outside or inside the conic according as the given pairs of points, A_1, A_2 and B_1, B_2, do not or do separate one another.

Corollary. *According as two given pairs of real points do not or do separate one another, the points separating both of them harmonically are real or conjugate-imaginary.*

EXERCISES

1. State and prove the dual of Theorem 1.

2. Discuss involutions among the tangents to a nondegenerate conic, proving the duals of Theorems 2–6 inclusive.

3. Deduce from Theorem 7 and its corollary the corresponding theorems for (*a*) lines through a point; (*b*) points on a line. See Ch. IX, § 7, Th. 8.

4. Show that the lines which join two points A and B on a nondegenerate conic to a third point P on the conic are separated harmonically by the tangent at P and the line joining P to the pole of the line AB.

5. Prove directly that a pencil of lines whose vertex is not on a nondegenerate conic cuts the conic in two projective ranges of points.

11. Involutions in the Plane. An involution in the plane, that is, an involutory collineation of the plane, interchanges in pairs those points of the plane which it does not leave fixed. If A_1, A_2 and B_1, B_2 are two of these pairs which are not collinear, the involution carries each of the lines A_1A_2 and B_1B_2 into itself, and hence leaves fixed the point O common to these lines.

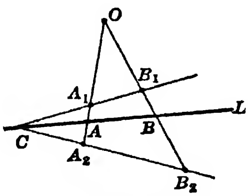


Fig. 24

Since A_1, A_2, and O are carried into A_2, A_1, and O, and B_1, B_2, and O into B_2, B_1, and O, the harmonic conjugates, A and B, of O with respect to A_1, A_2 and B_1, B_2, respectively, are carried into themselves. Their line L is therefore a fixed line.

Since the involution interchanges A_1, A_2 and also B_1, B_2, it interchanges the lines A_1B_1 and A_2B_2, and hence leaves their point of intersection C fixed. Inasmuch as $(O\,A,\ A_1A_2) = (O\,B,\ B_1B_2)$, C lies on the line L.

We now have on L three fixed points, A, B, C. The projective transformation of L into itself which is established on L by the involution is consequently the identity. Every point on L is a fixed point.

Because O and each point on L are fixed points, each line through O is a fixed line.

The point O and the points on L are fixed points and the line L and the lines through O are fixed lines. These are all the fixed points and fixed lines. If there were, for example, an additional fixed point, it would be possible to single out four fixed points, no three of which are collinear, and the collineation would be the identity, by Ex. 3, End of Ch. X.

The projective transformation established by the given involution on a fixed line through O is itself a (one-dimensional) involution. The fixed points on the line, namely O and the intersection of the line with L, are the fixed points of this involution, and the pairs of points separating them harmonically are the pairs in this involution. Thus:

The points of each pair of points in the involution are collinear with O and separate harmonically the point O and the intersection of their line with L. Dually, the lines of each pair of lines in the involution meet on L and separate harmonically the line L and the line joining their common point with O.

The point O is called the *center*, and the line L the *axis*, of the involution. Evidently:

THEOREM 1. *There is a unique involution which has a given point as center and a given line as axis, provided the point does not lie on the line.*

Since, in the original argument, the center O and the axis L were determined by the two pairs of points A_1, A_2 and B_1, B_2, we also have

THEOREM 2. *An involution is uniquely determined by two pairs of points, provided that no three of the four points are collinear.*

Relationship between Involutions in a Plane and Involutions on a Conic. An involution in a plane transforms into itself any nondegenerate conic with respect to which its center O and axis L are pole and polar. In particular, it interchanges each two points of the conic which are collinear with O. Thus it establishes in the points of the conic an involution. Conversely, if an involution in the points of a conic is given, there exists, as can be shown by application of Th. 2, a unique involution in the plane which leaves the conic fixed and establishes in the points of the conic the given involution.

EXERCISES

1. Establish the second half of the italicized statement preceding Theorem 1.

2. *Metric Involutions in the Plane.* If the axis of an involution in the metric plane is the line at infinity, the involution is the reflection of the plane in a finite point. If the axis is a finite line L and the center is the point at infinity

in the direction perpendicular to L, the involution is the reflection of the plane in the line L. Prove these propositions.

3. Show that there is a unique collineation of the plane which interchanges the points of each of two given pairs of points, provided that no three of the four points are collinear, and that this collineation is an involution.

4. Discuss the relationship between involutions among the tangents to a conic and involutions in a plane.

12. Parametric Representations of Conics. Let A_1, A_2, A_3 be the triangle of reference, and D the unit point of a system of projective point coordinates (Fig. 25). Consider the projective correspondence between the pencils of lines at A_1 and A_3 which is determined by the three pairs of corresponding lines $b_1 \leftrightarrow b_3$, $c_1 \leftrightarrow c_3$, $d_1 \leftrightarrow d_3$ shown in the figure. Introduce in the two pencils projective coordinates based respectively on b_1, c_1, d_1 and b_3, c_3, d_3, and denote both coordinates by t. The lines in the pencils with the same coordinate t are, then, corresponding lines. Their equations are

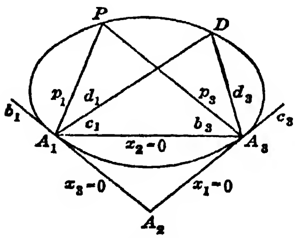


FIG. 25

$$(1) \qquad t\,x_3 - x_2 = 0, \qquad t\,x_2 - x_1 = 0.$$

Elimination of t from these equations gives the equation of the conic generated by the points of intersection of corresponding lines of the pencils (§ 5). Consequently, if we solve the equations for x_1, x_2, x_3 in terms of t we obtain a parametric representation of the conic:

$$(2) \qquad x_1 = t^2, \qquad x_2 = t, \qquad x_3 = 1.$$

Since there is only one type of conic with a real trace in projective geometry, equations (2) may be thought of as constituting a *normal form* for the parametric representation of a conic, obtained by a proper choice of the coordinate system and the parameter.

Equations (2) establish a perfect one-to-one correspondence between the values of t, including $t = \infty$, and the points of the conic. In other words, *t is a coordinate in the range of points on the conic.* But t is also a coordinate in the pencil of lines, say, at A_1. As a matter of fact, it is clear that a line L of the pencil and the point P in which it meets the conic have the same coordinate t. Hence we conclude:

Theorem 1. *If P_1, P_2, P_3, P_4 are four distinct points on the conic, with coordinates t_1, t_2, t_3, t_4, then*

$$(P_1P_2,\ P_3P_4) = \frac{(t_3 - t_1)(t_4 - t_2)}{(t_3 - t_2)(t_4 - t_1)}.$$

Theorem 2. *The projective transformations of the range of points on the conic into itself are identical with the linear transformations*

$$t' = \frac{a_1\,t + a_2}{b_1\,t + b_2}, \qquad a_1b_2 - a_2b_1 \neq 0. \tag{3}$$

Thus it is evident that the theory of projective correspondences between two ranges of points on the same conic, or on different conics, is analytically equivalent to the corresponding theory for linear one-dimensional fundamental forms.

The transformation (3) admits here the two usual interpretations. It may be thought of as representing, not only a projective transformation in the points of the conic, but also a change from one coordinate t on the conic to a second coordinate t'. The projective theory of points on the conic will have the same analytic form in terms of the new coordinate t' as it has been shown to have in terms of t. Accordingly, we shall call t and every new coordinate t' obtained from t by means of a linear transformation (3) *projective* coordinates on the conic.

From the point of view of the theory of parametric representation of curves, equation (3) or the equivalent equation

$$t = \frac{-\,b_2\,t' + a_2}{b_1\,t' - a_1}$$

represents a *change of parameter,* whereby the representation (2) becomes

$$\begin{aligned} x_1 &= (-\,b_2t' + a_2)^2, \\ x_2 &= (-\,b_2t' + a_2)(b_1t' - a_1), \\ x_3 &= (b_1t' - a_1)^2. \end{aligned} \tag{4}$$

The expressions on the right-hand sides of these equations are of the second degree in t'. We are thus led to investigate the general parametric equations of this type.

Theorem 3. *The equations*

$$\begin{aligned} x_1 &= a_{11}t^2 + a_{12}t + a_{13}, \\ x_2 &= a_{21}t^2 + a_{22}t + a_{23}, \\ x_3 &= a_{31}t^2 + a_{32}t + a_{33}, \end{aligned} \qquad \Delta = |a_{ij}| \neq 0, \tag{5}$$

represent a nondegenerate conic, and the parameter t is a projective coordinate on the conic.

Equivalent to (5) are the equations

$$\begin{aligned} A_{11}x_1 + A_{21}x_2 + A_{31}x_3 &= \Delta t^2, \\ A_{12}x_1 + A_{22}x_2 + A_{32}x_3 &= \Delta t, \\ A_{13}x_1 + A_{23}x_2 + A_{33}x_3 &= \Delta, \end{aligned}$$

and the change of coordinates

$$\Delta x'_i = \sum_j A_{ji}x_j, \qquad (i = 1,\ 2,\ 3)$$

reduces these equations to

$$x'_1 = t^2, \qquad x'_2 = t, \qquad x'_3 = 1.$$

Hence the theorem is proved.

Before leaving this subject, the student will find it profitable to read once more the paragraph on unicursal curves in Ch. XIII.

EXERCISES

1. The equation $t' = -t$ represents an involution in the points of the conic (2). Show that the center of the involution is A_2.

2. The equation $t' = k\,t$, $k^2 \neq 1$, represents a noninvolutory projective transformation of the points of the conic (2), with A_1 and A_3 as the double points. Show that the envelope of the lines joining corresponding points is a conic which is tangent to the given conic in A_1 and A_3.

3. Find the equation in point coordinates of the conic (5).

4. What do the equations (5) represent when $\Delta = 0$?

5. Develop the theory of parametric representations of the lines of a nondegenerate conic.

EXERCISES ON CHAPTER XV

1. The lines joining three fixed points A, B, C on a hyperbola to a variable point on the hyperbola cut a parallel to an asymptote in the points A', B', C'. Show that the ratio $A'B'/B'C'$ is constant.

2. Establish the corresponding proposition for a parabola.

3. Show that, if three fixed tangents to a parabola cut an arbitrary fourth tangent in the points A, B, C, then AB/BC is constant.

4. Prove that the triangle formed by the asymptotes and an arbitrary tangent to a hyperbola has a constant area.

Suggestion. Apply the second part of the dual of Steiner's Theorem to two tangents and the asymptotes.

5. Show that the locus of a point P which moves so that PA_1, PA_2 and PB_1, PB_2, where A_1, A_2 and B_1, B_2 are the pairs of opposite vertices of a parallelogram, form a harmonic set, is a central conic with the diagonals of the parallelogram as conjugate diameters.

6. Prove that the locus of a point which moves so that two lines with given directions through it are always separated harmonically by the lines joining it to two fixed points is a hyperbola. Locate its center and asymptotes.

7. The four sides of an ordinary quadrilateral turn about fixed points and three of the vertices trace fixed lines. Find the locus of the remaining vertex.

8. The base of a triangle is fixed and the two sides move so that they always intercept the same distance upon a fixed line. What is the locus of the vertex?

9. The vertices of a triangle trace three fixed concurrent lines and two sides turn about fixed points. What is the envelope of the third side?

10. A nondegenerate conic and two points P_1, P_2 not conjugate with respect to it are given. Find the locus of a point P which moves always so that P_1P and P_2P are conjugate lines.

11. The base of a triangle always touches a given nondegenerate conic, the extremities of the base move on two fixed tangents to the conic, and the two sides turn about fixed points. Find the locus of the vertex.

12. Two men are walking at constant rates, not necessarily equal, along intersecting straight roads. At a certain time a tree is in line with the men. Are there other times at which the tree and the men are collinear? Discuss all possibilities.

13. A point P traces a fixed line L. The point Q is the harmonic conjugate of P with respect to two fixed points on L, and the point P' is the projection of Q from a fixed point O on a second fixed line L'. Discuss the envelope of the line PP'. What metric theorem results when L is the line at infinity and the fixed points on it are the circular points?

14. The sides 1 and 4 of the hexagon inscribed in the conic in Fig. 12, p. 260, remain fixed while the sides 2 and 5 turn about B and E in such a manner that their point of intersection M traces a straight line. Discuss in detail the locus of the point N in which the sides 3 and 6 intersect.

15. A fixed line M and a fixed point F, not on M, are given. A point P traces M, and a line L through P moves always so that the directed angle from PF to L is constant. Prove that the envelope of L is a parabola with F as focus.

16. A nondegenerate conic is determined by five points. Show how to construct (*a*) the second point of intersection of a chord through one of the five points parallel to the chord determined by two others; (*b*) the center of the conic, if there is a center; (*c*) the polar of a point.

17. A nondegenerate conic is determined by five tangents. Construct (*a*) a sixth tangent parallel to one of the five; (*b*) the center, if there is a center.

18. Construct a line parallel to the axis of a nondegenerate parabola which is determined by four finite tangents.

19. A nondegenerate hyperbola is determined by four points and a line parallel to an asymptote. Show how to construct (*a*) this asymptote; (*b*) a line parallel to the second asymptote.

20. A parabola is determined by a line parallel to the axis, two finite tangents and the contact point on one of them. Show how to construct other tangents.

21. A conic is inscribed in a triangle and is tangent to two of the sides at their mid-points. Show that it is also tangent to the third side at its mid-point.

22. An arbitrary nondegenerate conic is passed through four points O, A, B, C, no three of which are collinear. Fixed lines L_1 and L_2 through O meet the conic again in P_1 and P_2. Show that the Pascal line of the hexagon OP_1ABCP_2O is independent of the conic chosen.

23. Discuss the envelope of the line P_1P_2 of the preceding exercise. Suggestion: Consider the motion of the triangle P_1P_2M, where M is the intersection of P_1A and P_2C.

24. Prove by means of Brianchon's Theorem that the altitudes of a triangle formed by three finite tangents to a parabola intersect on the directrix.

Suggestion: Form the circumscribed hexagon from the three given tangents, the tangents perpendicular to two of them, and the line at infinity.

25. Show that, if two triangles with distinct vertices are inscribed in a nondegenerate conic, their sides are tangent to a nondegenerate conic.

26. The base of a triangle turns about a fixed point, the extremities of the base move on a nondegenerate conic, and the sides turn about fixed points on the conic. What is the locus of the vertex?

27. A nonparabolic projective correspondence between two ranges of points on the same conic is given. Show that the envelope of the lines joining corresponding points of the two ranges is a conic which is tangent to the given conic at each of the double points of the correspondence. State the dual theorem.

28. A variable triangle is inscribed in a nondegenerate conic and two of its sides turn about fixed points. What is the envelope of the third side?

29. Show that, if a collineation carries a nondegenerate conic Q_1 into a second conic Q_2, it establishes between the ranges of points on Q_1 and Q_2 a projective correspondence.

30. Prove that there exists a unique collineation of the plane which carries a given nondegenerate conic into a prescribed nondegenerate conic and establishes between the ranges of points on the two conics a given projective correspondence. What does the proposition become when the second conic is taken as identical with the first?

CHAPTER XVI

PAIRS AND PENCILS OF CONICS

A. Point Conics

1. Pairs of Point Conics. Geometrical Discussion. In how many points do two distinct point conics intersect? Our experience tells us, four. Certainly not more, in general, for two nondegenerate conics which have five points in common are identical.

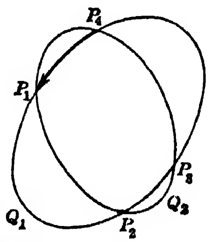


FIG. 1

I. Consider two nondegenerate conics, Q_1 and Q_2, and assume that they do intersect in four distinct points P_1, P_2, P_3, P_4. They cannot be tangent at any one of the four points, for if they were tangent at P_1, for example, we should have two distinct conics tangent at P_1 to a given line and passing through P_2, P_3, P_4, and this is impossible.

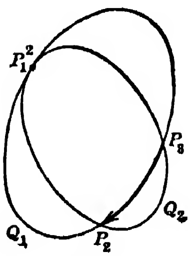


FIG. 2

Two circles which have two finite points in common intersect in four distinct points, the two finite points and the circular points at infinity.

II. Hold the conic Q_1 in Fig. 1 fast and vary the conic Q_2 so that its intersections P_1, P_2, P_3 with Q_1 remain fixed, while the fourth intersection P_4 approaches P_1 as a limit. Since P_4 approaches P_1 along Q_1, the limit of the line P_1P_4 is the tangent to Q_1 at P_1. It is also the tangent at P_1 to the conic which is the limit of Q_2, since it intersects this conic only in P_1. Hence Q_1 and the new conic Q_2 are tangent at the *doubly counting* point of intersection, P_1, and intersect in P_2 and P_3 (Fig. 2); that they are not tangent at either P_2 or P_3 follows from Ch. XV, § 4, Th. 3 *a*.

An example of this type is furnished by two circles which are tangent at a finite point.

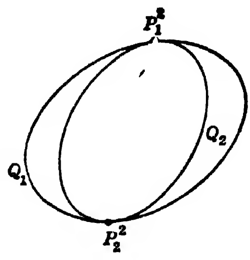


FIG. 3

III. If in Fig. 2 we allow Q_2 to approach a new position in such a way that the intersection

P_3 with Q_1 approaches the intersection P_2 as a limit, we obtain two conics tangent to one another in each of two points (Fig. 3).

Two concentric circles are tangent to one another at each of the circular points at infinity. Why?

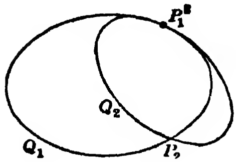


Fig. 4

IV. If, instead of allowing P_3 in Fig. 2 to approach P_2, we allow it to approach P_1, we obtain two conics tangent at a point which counts *three* times as a point of intersection (Fig. 4). The other point of intersection P_2 is not a point of tangency, as will be proved presently.

V. Finally, we allow P_2 in Fig. 4 to approach P_1 as a limit. The four points common to Q_1 and Q_2 then coincide in one *quadruple* point of intersection (Fig. 5).

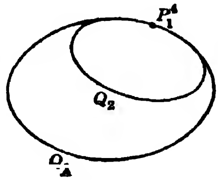


Fig. 5

EXERCISES

1. Discuss in detail the points of intersection of a nondegenerate and a degenerate point conic, drawing figures to represent the various cases possible. Show that all five types of intersection actually occur.

2. The same for two degenerate point conics which do not have a line in common. Only four of the five types occur. Which one is missing?

2. Continuation. Analytic Proof. To establish beyond question the facts brought out in the previous paragraph, we proceed analytically, beginning with a careful choice of the basic points A_1, A_2, A_3, D of our system of projective coordinates. Let A_2 be a point on Q_1 not also on Q_2, A_3 a second point on Q_1, A_1 the intersection of the tangents to Q_1 at A_2 and A_3, and D a point on Q_1.

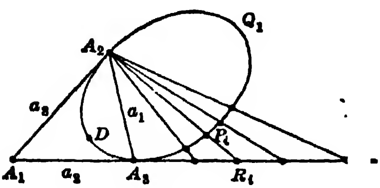


Fig. 6

The equation of Q_1 is then

$$x_2x_3 - x_1^2 = 0,$$

while that of Q_2 is

$$ax_1^2 + bx_1x_2 + cx_2^2 + dx_1x_3 + ex_2x_3 + fx_3^2 = 0,$$

where, since A_2 is not on Q_2, $c \neq 0$.

Since a_3 meets Q_1 only at A_2 and A_2 is not on Q_2, Q_1 and Q_2 cannot intersect on a_3. Hence we may assume, without losing points of intersection, that $x_3 \neq 0$ and introduce $x = x_1/x_3$, $y = x_2/x_3$ in place of x_1, x_2, x_3. The equations of Q_1 and Q_2 become

$$(1) \qquad y = x^2,$$

$$(2) \qquad ax^2 + bxy + cy^2 + dx + ey + f = 0, \qquad c \neq 0.$$

Our problem is to solve these equations simultaneously. What is simpler? The substitution of the value of y from (1) into (2) yields the equation

$$(3) \qquad cx^4 + bx^3 + (a + e)x^2 + dx + f = 0, \qquad c \neq 0.$$

Since $c \neq 0$, this equation has four roots r_1, r_2, r_3, r_4. Consequently, Q_1 and Q_2 have in common the four points

$$P_i: \qquad (r_i, r_i^2, 1), \qquad (i = 1, 2, 3, 4).$$

The solutions r_1, r_2, r_3, r_4 of (3) determine on a_2 the four points

$$R_i: \qquad (r_i, 0, 1), \qquad (i = 1, 2, 3, 4).$$

The line P_iR_i goes through A_2, for it has the equation $x_1 = r_ix_3$. Hence the point R_i is the projection from A_2 of the point P_i, and vice versa. Thus our method of solving equations (1) and (2), interpreted geometrically, consists in projecting the possible points of intersection of Q_2 with Q_1 from A_2 on a_2, in discovering four projected points on a_2 and projecting them back on Q_1.

There are five possibilities as to the roots of equation (3): four simple roots; two simple roots and one double root; two double roots; one simple root and one triple root; one quadruple root. In each case, *the sum of the number of times which each distinct root counts is equal to four*, and this is what we mean when we say that the equation has four roots.

Since the roots r_i of (3) are not necessarily distinct, neither are the points R_i nor the points P_i. If certain of the points R_i are identical, the corresponding points P_i are identical, and conversely. Thus a point P counts as many times as a point of intersection of Q_1 and Q_2 as does the corresponding r as a root of (3). Consequently, there are five possibilities: four simple points of intersection, two simple points and one double point, two double points, one simple and one triple point, and, finally, one quadruple point. In every case the sum of the number of times each distinct point counts is four. Hence:

THEOREM 1. *Two distinct nondegenerate conics intersect always in four points.*

To justify completely the expectations of § 1, we establish the following theorem.

THEOREM 2. *The two conics are never tangent at a simple point of intersection and are always tangent at a multiple point of intersection.*

In proving the theorem, we take, as the point A_3, a point of intersection of Q_2 with Q_1. Then $f = 0$ and equations (2) and (3) become

$$ax^2 + bxy + cy^2 + dx + ey = 0, \tag{2'}$$

$$cx^4 + bx^3 + (a + e)x^2 + dx = 0, \qquad c \neq 0. \tag{3'}$$

The point $A_3 : x = 0,\ y = 0$ is a simple or multiple point of intersection of Q_1 and Q_2 according as $x = 0$ is a simple or multiple root of (3′), that is, according as $d \neq 0$ or $d = 0$. On the other hand, the tangents at A_3 to Q_1 and Q_2, namely

$$y = 0 \qquad dx + ey = 0,\text{*}$$

are distinct or identical according as $d \neq 0$ or $d = 0$. Thus the theorem is established.†

A multiple point of intersection may be a double, a triple, or a quadruple point. Accordingly, we recognize three types of tangency.

DEFINITION. *If two nondegenerate conics are tangent at a point P, they are said to have at P two-point contact, three-point contact, or four-point contact according as P is a double, triple, or quadruple point of intersection. If the two conics have two-point contact at each of two points, they are said to have double contact at these points.*

We have now established the theorems and formulated the definitions which are necessary for accurate descriptions of the five possible types of intersection.

I: $[P_1, P_2, P_3, P_4]$.‡ Q_1 and Q_2 intersect *simply*, that is, without being tangent, at each of four distinct points.

* Since Q_2 is nondegenerate, d and e cannot both be zero.

† The method of proof of Theorems 1, 2 implies that Q_1, and Q_2 have real traces and a real point of intersection. However, the method applies equally well to complex conics and in this case puts no restrictions on the conics, since in complex geometry a triangle does not have to have real vertices in order to serve as a triangle of reference for projective coordinates.

‡ This symbol denotes four simple points of intersection; similarly, $[P_1^3, P_2]$ means one triple and one simple point of intersection.

II: $[P_1^2, P_2, P_3]$. Q_1 and Q_2 have two-point contact at P_1 and intersect simply in each of the points P_2, P_3.

III: $[P_1^2, P_2^2]$. Q_1 and Q_2 have double contact at the points P_1 and P_2.

IV: $[P_1^3, P_2]$. Q_1 and Q_2 have three-point contact at P_1 and intersect simply at P_2.

V: $[P_1^4]$. Q_1 and Q_2 have four-point contact at P_1.

Examples of the first three types were given in § 1 and examples of the last two types are readily constructed.* All five types, then, actually exist.

Suppose that one of the given point conics, say Q_2, is degenerate. Then each line of Q_2 intersects Q_1 in two points and hence Q_2 intersects Q_1 in four points. If both conics are degenerate and have not a line in common, each line of Q_1 intersects each line of Q_2 in a single point and hence here, too, there are four points of intersection.

THEOREM 3. *Two distinct point conics which have not a line in common intersect always in four points.*

Guided by Theorem 2, we agree to say that two point conics, one or both of which are degenerate, are tangent at a common point P if P is a multiple point of intersection.† We agree also to apply the definition of the three types of tangency to the present case. This means, for example, that the two conics in Fig. 7 a have three-point contact at P_1, and that those in Fig. 7 b have double contact at P_1 and P_2.

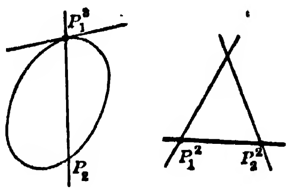


FIG. 7 a FIG. 7 b

The following proposition will be found useful later.

THEOREM 4. *If two nondegenerate conics Q_2 and Q_2' have k-point contact with a nondegenerate conic Q_1 at a point P, they have at least k-point contact with each other at P.*

* The conics (1) and (2′) have three-point contact at A_3 if $x = 0$ is a triple root of (3′), that is, if $d = 0$, $a + e = 0$, $b \neq 0$. Hence an example of Type IV is

$$y = x^2, \qquad a(x^2 - y) + bxy + cy^2 = 0, \qquad abc \neq 0.$$

† Note that, in the case of a nondegenerate conic Q_1 and a degenerate conic Q_2, we are specifying, not the conditions under which the separate lines of Q_2 are tangent to Q_1—that has already been done—but the conditions under which Q_2 as a whole is tangent to Q_1.

It is evident that, if Q_2 and Q'_2 are tangent to Q_1 at P, they are tangent to one another at P. Hence the theorem is true when $k = 2$.

In giving the proof when $k = 3$, we take as Q_1 the conic $x^2 - y = 0$ and as P the point $A_3 : x = 0,\ y = 0$. Then, according to the first footnote on p. 283,

$$Q_2: \qquad a(x^2 - y) + bxy + cy^2 = 0, \qquad a\,b\,c \neq 0,$$

$$Q'_2: \qquad a'(x^2 - y) + b'xy + c'y^2 = 0, \qquad a'b'c' \neq 0,$$

are two arbitrary nondegenerate conics having three-point contact with Q_1 at A_3. That they have at least three-point contact with one another at A_3 follows from the fact, which we leave to the reader to verify, that, of the four simultaneous solutions of their two equations, at least three are $x = 0,\ y = 0$.

A similar proof can be given when $k = 4$.

EXERCISES

1. Find the points of intersection of each of the following pairs of conics.

(a) $x^2 = y + 1, \qquad 2\,x^2 + 2\,xy + y^2 - y - 2 = 0;$

(b) $y^2 - 4\,x = 0, \qquad 16\,x^2 + 16\,y^2 - 40\,x + 9 = 0;$

(c) $xy - x - y = 0, \qquad x^2 + xy - 4\,y = 0;$

(d) $xy = 1, \qquad (x - 1)\,y = 1.$

2. Construct an example of a pair of nondegenerate conics of Type V. Then prove Theorem 4 when $k = 4$.

3. Pencils of Point Conics. General Case.

If

$$\rho \equiv \sum a_{ij}x_ix_j = 0, \qquad \sigma \equiv \sum b_{ij}x_ix_j = 0$$

are two distinct point conics, the totality of point conics represented by the equation

$$(1) \qquad k\,\rho + l\,\sigma = 0,$$

when k and l take on all possible pairs of values other than 0, 0, is called a *pencil of point conics*. The conics $\rho = 0$ and $\sigma = 0$ are known as the *base conics* of the pencil.

We consider, first, the general case in which $\rho = 0$ and $\sigma = 0$ are nondegenerate conics which intersect in four distinct points P_1, P_2, P_3, P_4.

In this case, *the pencil consists precisely of all the point conics which pass through the four points.* In the first place, every conic of the pencil goes through the four points; for, if one of the points is (r_1, r_2, r_3),

then $\rho(r_1, r_2, r_3) = 0$, $\sigma(r_1, r_2, r_3) = 0$ and the equation

$$k\,\rho(r_1, r_2, r_3) + l\,\sigma(r_1, r_2, r_3) = 0$$

is true, no matter what the values of k and l are. Conversely, let C be a conic through the four points, and let $P:(s_1, s_2, s_3)$ be a fifth point on C. Since C is uniquely determined by five points, it will follow that C belongs to the pencil if we can exhibit a conic of the pencil which passes through P. But such a conic is

$$\sigma(s_1, s_2, s_3)\,\rho - \rho(s_1, s_2, s_3)\,\sigma = 0.$$

It is evident that a pair of opposite sides of the complete quadrangle determined by the four points constitutes a degenerate point conic of the pencil, and conversely. Hence the pencil contains three distinct degenerate point conics.

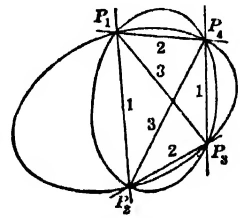


FIG. 8

Since the pencil is the totality of conics which pass through the points P_1, P_2, P_3, P_4, and since each two of the conics intersect in these points, any two distinct conics of the pencil may be employed as the base conics instead of $\rho = 0$, $\sigma = 0$. In particular, we may take as the base conics two of the degenerate conics. Thus, if $\alpha = 0, \beta = 0$ are the equations of the lines P_1P_2, P_3P_4, and $\gamma = 0$, $\delta = 0$, those of the lines P_1P_4, P_2P_3, we obtain as a new equation of the pencil

$$k\,\alpha\beta + l\,\gamma\delta = 0. \tag{2}$$

If four points R_1, R_2, R_3, R_4, no three collinear, are given, the point conics which pass through them constitute a pencil. If $\alpha = 0, \beta = 0$, $\gamma = 0$, $\delta = 0$ are equations of the lines R_1R_2, R_3R_4, R_1R_4, R_2R_3, then $\alpha\beta = 0$ and $\gamma\delta = 0$ are two degenerate conics of the pencil and (2) is an equation of the pencil.*

Conic through Five Points. It is now absurdly simple to write the equation of the conic which passes through five given points. We have only to write equation (2) of the general conic passing through

* If r', r'', r''', r^{iv} are the symbolic coordinates of the four points, the explicit equation is

$$k\,(x\,r'r'')(x\,r'''r^{\text{iv}}) + l\,(x\,r'r^{\text{iv}})(x\,r''r''') = 0,$$

where $(x\,r'r'')$, for example, is the determinant $|\,x\,r'r''\,|$.

four of the points and determine k and l so that this conic goes through the fifth point.*

The method may readily be applied to the following affine and metric problems: to find (a) the central conic through four given points with an asymptote of given slope; (b) the central conic through three given points with given slopes for both asymptotes, in particular, the circle through three points.

EXERCISES

1. Find the equation of the conic through the five points:

(a) $(1, 0)$, $(-1, 0)$, $(0, 1)$, $(0, -1,)$, $(1, 1)$;

(b) $(0, 0)$, $(2, 1)$, $(3, -4)$, $(0, 2)$, $(-2, 0)$.

2. Find the equation of the hyperbola which goes through the points $(1, 1)$, $(-1, -1)$, $(1, -1)$, $(-1, 1)$ and has an asymptote of slope $1/3$.

3. The same, if the hyperbola goes through the points $(1, 0)$, $(0, 1)$, $(1, 1)$ and has asymptotes of slopes $1/2$ and 2.

4. Find the equation of the circle through the points $(1, 2)$, $(-2, 1)$, $(-2, -4)$.

4. Continuation. Systematic Treatment. A pencil of point conics is the totality of point conics

(1) $$k\,\rho + l\,\sigma = 0,$$

which are linearly dependent on two distinct point conics,

(2) $$\rho \equiv \sum a_{ij}x_ix_j = 0, \qquad \sigma \equiv \sum b_{ij}x_ix_j = 0.$$

Two distinct conics of the pencil are

(3) $$\rho_1 \equiv k_1\rho + l_1\sigma = 0, \qquad \rho_2 \equiv k_2\rho + l_2\sigma = 0,$$

where

$$\Delta = k_1l_2 - k_2l_1 \neq 0.$$

A conic linearly dependent on $\rho_1 = 0$, $\rho_2 = 0$ is also linearly dependent on $\rho = 0$, $\sigma = 0$, for

$$m\,\rho_1 + n\,\rho_2 \equiv (mk_1 + nk_2)\,\rho + (ml_1 + nl_2)\,\sigma.$$

Conversely, a conic linearly dependent on $\rho = 0$, $\sigma = 0$ is also linearly

* If the five points are r', r'', r''', r^{iv}, r^{v} the equation of the conic is

$$(r'r^{\text{iv}}r^{\text{v}})(r''r'''r^{\text{v}})(x\,r'r'')(x\,r'''r^{\text{iv}}) - (r'r''r^{\text{v}})(r'''r^{\text{iv}}r^{\text{v}})(x\,r'r^{\text{iv}})(x\,r''r''') = 0.$$

If it is assumed, as is necessary in order that the problem have meaning, that no four of the five points are collinear, it is always possible to choose from them four points, no three collinear, as r', r'', r''', r^{iv}.

dependent on $\rho_1 = 0$, $\rho_2 = 0$, for from (3)

$$\Delta\rho \equiv l_2\rho_1 - l_1\rho_2, \qquad \Delta\sigma \equiv -k_2\rho_1 + k_1\rho_2,$$

and hence

$$\Delta(k\,\rho + l\,\sigma) \equiv (kl_2 - lk_2)\,\rho_1 - (kl_1 - lk_1)\,\rho_2.$$

Thus, the pencil of conics determined by $\rho_1 = 0$, $\rho_2 = 0$ is identical with the given pencil. In other words:

THEOREM 1. *Any two distinct conics of a pencil may be taken as the base conics.*

The conic (1) contains the point r if k and l are so chosen that

$$k\,\rho(r_1, r_2, r_3) + l\,\rho(r_1, r_2, r_3) = 0.$$

Hence, we conclude

THEOREM 2. *Through each point of the plane other than the points common to all the conics of a pencil there passes a unique conic of the pencil.*

The conic (1),

$$k\,\rho + l\,\sigma \equiv \sum(k\,a_{ij} + l\,b_{ij})x_ix_j = 0,$$

is degenerate if and only if its discriminant vanishes:

$$\begin{vmatrix} k\,a_{11} + l\,b_{11} & k\,a_{12} + l\,b_{12} & k\,a_{13} + l\,b_{13} \\ k\,a_{21} + l\,b_{21} & k\,a_{22} + l\,b_{22} & k\,a_{23} + l\,b_{23} \\ k\,a_{31} + l\,b_{31} & k\,a_{32} + l\,b_{32} & k\,a_{33} + l\,b_{33} \end{vmatrix} = 0.$$

This equation becomes

$$|a_{ij}|\,k^3 + S_1\,k^2l + S_2\,kl^2 + |b_{ij}|\,l^3 = 0, \tag{4}$$

where $|a_{ij}|$, $|b_{ij}|$ are the discriminants of $\rho = 0$, $\sigma = 0$ and S_1 and S_2 are constants depending on the a's and b's. Unless all of its coefficients are zero—a case which we exclude here (see Ex. 11)—the equation has three roots. Hence:

THEOREM 3. *A pencil of point conics contains three degenerate point conics.*

THEOREM 4. *The points of intersection of two conics of a pencil and the number of times each point counts are the same for each two conics of the pencil.*

To prove Theorem 4, we shall first show that the points of intersection of $\rho_1 = 0$, $\rho_2 = 0$ are the same as those of $\rho = 0$, $\sigma = 0$. The

point r is common to $\rho = 0$, $\sigma = 0$ if and only if

$$(5) \qquad \rho(r_1, r_2, r_3) = 0, \qquad \sigma(r_1, r_2, r_3) = 0,$$

and common to $\rho_1 = 0$, $\rho_2 = 0$ if and only if

$$(6) \quad k_1\rho(r_1, r_2, r_3) + l_1\sigma(r_1, r_2, r_3) = 0, \quad k_2\rho(r_1, r_2, r_3) + l_2\sigma(r_1, r_2, r_3) = 0.$$

That equations (6) follow from equations (5) is obvious. Vice versa, since $k_1l_2 - k_2l_1 \neq 0$, equations (5) follow from equations (6). Hence the points common to the one pair of conics are identical with those common to the other pair.

Secondly, we shall prove that a point is a simple or multiple point of intersection of $\rho_1 = 0$, $\rho_2 = 0$ according as it is a simple or multiple point of intersection of $\rho = 0$, $\sigma = 0$. For this purpose we shall assume that $\rho = 0$, $\sigma = 0$ are the nondegenerate conics (1) and (2′) of § 2, namely *

$$\rho \equiv y - x^2 = 0, \quad \sigma \equiv ax^2 + bxy + cy^2 + dx + ey = 0,$$

and that the point of intersection is $A_3 : x = 0, y = 0$. By § 2, Th. 2, the point A_3 is a simple or multiple point of intersection of $\rho = 0, \sigma = 0$ according as the tangents at A_3 to $\rho = 0$, $\sigma = 0$, namely

$$y = 0, \qquad dx + ey = 0,$$

are distinct or coincident. Similarly, A_3 is a simple or multiple point of intersection of $\rho_1 = 0$, $\rho_2 = 0$ according as the tangents at A_3 to $\rho_1 = 0$, $\rho_2 = 0$, whose equations are readily found to be

$$k_1y + l_1(dx + ey) = 0, \qquad k_2y + l_2(dx + ey) = 0,$$

are distinct or coincident. But the latter tangents are evidently distinct or coincident according as the former are distinct or coincident. Hence the contention is established.†

Theorem 4 may now be readily proved by inspection of the various possibilities. Suppose, for example, that $\rho = 0$, $\sigma = 0$ have two dis-

* Since the pencil contains infinitely many nondegenerate conics (Th. 3), and any two of its conics may be taken as the base conics (Th. 1), we may assume that $\rho = 0$, $\sigma = 0$ are nondegenerate conics. Their equations may then be reduced to the forms in question by the method of § 2.

† The argument fails if the equation given for the tangent at A_3 to, say, $\rho_1 = 0$ is illusory. But, since the equation $\sum a_{ij}r_ix_j = 0$ is illusory only if r is a singular point of the conic $\sum a_{ij}x_ix_j = 0$, $\rho_1 = 0$ must, in this case, consist of two lines through A_3 and hence must intersect $\rho_2 = 0$ at least twice in A_3. On the other hand, if the equation $k_1y + l_1(dx + ey) = 0$ is illusory, then $d = 0$ and $\rho = 0$, $\sigma = 0$ intersect at least twice in A_3. Hence the contention is also valid in this case.

tinct points of intersection, one of which is a simple point and the other a triple point. Then $\rho_1 = 0$, $\rho_2 = 0$ must have, as their intersections, the same simple point and the same multiple point and, since the total number of intersections is four, the multiple point must be a triple point.

It is evident from Theorem 4 that there are five different types of pencils of point conics, corresponding to the five types of intersection of two conics.

I: $[P_1, P_2, P_3, P_4]$. This is the general case which we discussed in detail in § 3.

THEOREM I. *The pencil of conics consists of all the point conics passing through four points, no three of which are collinear.*

The three degenerate conics are distinct, each consisting of a pair of opposite sides of the complete quadrangle $P_1P_2P_3P_4$.

II: $[P_1^2, P_2, P_3]$. Each two conics of the pencil, since they intersect twice in P_1, are tangent at P_1. Consequently, all the conics of the pencil have the same tangent t_1 at P_1.

THEOREM II. *The pencil consists of all the point conics which are tangent to a given line t_1 at a given point P_1 and pass through two other points P_2, P_3.**

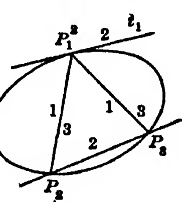


FIG. 9

For, every conic of the pencil has these properties, and every conic which has them is determined by them and a further point, and hence, by Th. 2, belongs to the pencil.

The pencil contains two distinct degenerate conics, namely t_1, P_2P_3 and P_1P_2, P_1P_3. The latter counts twice, as may be verified by considering the pencil as a limiting case of a pencil of Type I.

By taking the degenerate conics as the base conics, we obtain a simple equation representing the pencil. Let the equations of t_1 and P_2P_3 be $\tau_1 = 0$ and $\beta = 0$, and let those of P_1P_2 and P_1P_3 be $\gamma = 0$ and $\delta = 0$. The desired equation is then

$$k\,\tau_1\beta + l\,\gamma\delta = 0. \tag{7}$$

A simple method now presents itself for writing the equation of a conic tangent to a given line t_1 at a given point P_1 and passing through

* In order that the degenerate conic P_1P_2, P_1P_3 be covered by this description we must recognize a line through its singular point as a tangent.

three other given points P_2, P_3, P_4. The general conic tangent to t_1 at P_1 and going through P_2 and P_3 has the equation (7), and the constants k, l in this equation may be determined so that the conic contains P_4.

The method is applicable to various affine and metric problems, for example, to that of finding the parabola through three points with an axis of given slope.

III: $[P_1^2, P_2^2]$. Each two conics have double contact at P_1 and P_2. Hence all the conics have the same tangents t_1 and t_2 at P_1 and P_2. Conversely, every conic which is tangent to t_1 at P_1 and to t_2 at P_2 is determined by a further point and therefore, by Th. 2, belongs to the pencil.

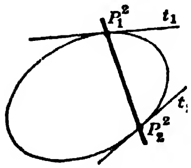


FIG. 10

THEOREM III. *The pencil consists of all the point conics which are tangent to each of two given lines t_1 and t_2 at given points P_1 and P_2.**

Here, too, there are two distinct degenerate conics in the pencil. The tangents t_1, t_2 form the one, and the chord of contact taken twice constitutes the other. If t_1, t_2 have the equations $\tau_1 = 0$, $\tau_2 = 0$ and P_1P_2 has the equation $\gamma = 0$, then

(8) $$k\,\tau_1\tau_2 + l\,\gamma^2 = 0$$

represents the pencil.

It is easy now to write the equation of the conic tangent to two lines at given points and passing through a third point.

IV: $[P_1^3, P_2]$. Each two conics of the pencil have three-point contact at P_1 and hence all have the same tangent t_1 at P_1. There is just one (triply counting) degenerate conic, namely t_1, P_1P_2. If $\rho = 0$ is a nondegenerate conic of the pencil and $\tau_1 = 0$ and $\beta = 0$ are the equations of t_1 and P_1P_2, a new equation of the pencil is

(9) $$k\,\rho + l\,\tau_1\beta = 0.$$

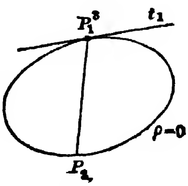


FIG. 11

THEOREM IV. *The pencil consists of all the point conics which have three-point contact with a given nondegenerate conic at a given point P_1 and intersect this conic simply in a second point P_2.*

* What agreement must be made in order that the degenerate conic consisting of the line P_1P_2 taken twice be covered by this description?

It is evident that the conics of the pencil answer this description. It remains to show that, if a nondegenerate conic C has three-point contact with $\rho = 0$ at P_1 and passes through P_2, it belongs to the pencil. Let P_3 be a further point on C, and let C' be the conic of the pencil which passes through P_3 (Th. 2). Since C and C' each have three-point contact with $\rho = 0$ at P_1, they have at least three-point contact with one another at P_1; see § 2, Th. 4. But they intersect also in P_2, P_3 and so have at least five points in common. Consequently, C coincides with C' and hence belongs to the pencil.

V: $[P_1^4]$. Each two conics of the pencil have four-point contact at P_1 and therefore all have the same tangent t_1 at P_1. This tangent counted twice constitutes the only degenerate conic in the pencil. A new equation of the pencil is

$$k\,\rho + l\,\tau_1^2 = 0, \tag{10}$$

where $\rho = 0$ is a chosen nondegenerate conic of the pencil and $\tau_1 = 0$ is the equation of t_1.

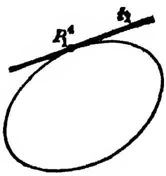


FIG. 12

THEOREM V. *The pencil consists of all the point conics which have four-point contact at a given point with a given nondegenerate conic.*

The proof of this theorem is similar to that of Theorem IV.

EXERCISES

Find the equations of the following conics.

1. The conic which is tangent to the line $x_1 + x_2 - x_3 = 0$ at the point $(0, 1, 1)$ and passes through the points $(-1, 0, 1)$, $(-1, 1, 1)$, $(0, 0, 1)$.

2. The parabola which has an axis of slope unity and goes through the points $(0, 0)$, $(1, 0)$, $(0, 1)$.

3. The conic tangent to $x + y = 1$ at $(1, 0)$ and to $x - y = 0$ at $(0, 0)$, and passing through $(3, 2)$.

4. The conic which has three-point contact with $2x^2 + y^2 + x - 1 = 0$ at the point $(0, 1)$, meets this conic again in the point $(-1, 0)$, and passes through the origin.

5. All the hyperbolas which have the lines $x - y = 0$, $x + y - 2 = 0$ as asymptotes.

6. The hyperbola which goes through the points $(1, 0)$, $(0, 1)$, $(0, 4)$ and has the line $4x - 2y + 5 = 0$ as an asymptote.

7. The hyperbola which goes through the point $(1, 3)$, is tangent to the y-axis at $(0, 4)$, and has the line $x - y - 2 = 0$ as an asymptote.

8. Show that *a pencil of point conics contains in general* * *just two parabolas.*

9. Find the equations of the parabolas which pass through the four points (0, 2), (0, − 2), (1, 2), (4, − 4).

10. Find the equations of the degenerate conics of the pencils determined by the pairs of conics of § 2, Ex. 1.

11. Prove that, if two degenerate point conics have a line in common or a singular point in common, the pencil of conics determined by them consists entirely of degenerate conics. Show that these are the only cases in which all the conics of a pencil are degenerate.

5. Conditions and Degrees of Freedom. A demand upon a conic which is equivalent to one analytic condition on the coefficients in the equation of the conic is known as *one condition* on the conic. The demand that a conic go through a given point places one condition on it, and the demand that it go through three points puts on it three conditions.

The general conic depends on five independent parameters, or has, as we say, *five degrees of freedom.* If three *independent* conditions are placed on the conic, three independent relations subsist between the five parameters. There are, then, ∞^2 conics satisfying the three conditions; or, the general conic satisfying them has two degrees of freedom.

Demanding that a conic be tangent to a given line at a given point puts two independent conditions on the conic, for there still remain three degrees of freedom—we may still demand that the conic go through three further points.

Again, the fact that there are ∞^3 circles in the plane is a direct corollary to the fact that a conic is a circle if and only if it passes through the two circular points.

6. Conics Tangent to a Given Conic at a Given Point. Let it be required to determine the number and the equation of the point conics which have k-point contact with a given nondegenerate conic $\rho = 0$ at a given point P.

The theory of pencils suggests that, if we find the most general degenerate point conic which has k-point contact with $\rho = 0$ at P, the conics which are linear combinations of $\rho = 0$ and this degenerate conic will constitute all the required conics. To establish the validity of this claim, it suffices to show that an arbitrarily chosen point conic having k-point contact with $\rho = 0$ at P is linearly dependent on $\rho = 0$

* The exceptions are discussed in § 8, Ex. 9.

and *some* degenerate point conic having k-point contact with $\rho = 0$ at P. This is readily proved. We have merely to take, as the degenerate conic, a degenerate conic of the pencil determined by the chosen conic and $\rho = 0$.

Four-Point Contact. There is just one degenerate point conic which has four-point contact with $\rho = 0$ at P, namely the tangent $\tau = 0$ at P counted twice. Hence all the required conics are given by

$$m\,\rho + \tau^2 = 0, \tag{1}$$

and form with $\rho = 0$ a pencil of Type V.

THEOREM 1. *There are ∞^1 point conics which have four-point contact with $\rho = 0$ at P.*

Application of Th. 2 of § 4 yields the following important theorem.

THEOREM 2. *There is a unique conic which has four-point contact with a given nondegenerate conic at a given point and passes through a second given point, not on the given conic.*

Three-Point Contact. If $\alpha = 0$ is an arbitrary line through P other than $\tau = 0$, the most general degenerate point conic which has three-point contact with $\rho = 0$ at P is $\alpha\tau = 0$, and the required conics are

$$m\,\rho + \alpha\tau = 0. \tag{2}$$

This equation contains two independent arbitrary constants. One of them is m and the other is contained in $\alpha = 0$.

THEOREM 3. *There are ∞^2 point conics which have three-point contact with $\rho = 0$ at P.*

If $\alpha = 0$ is allowed to take on the position of $\tau = 0$, the conics (2) include the conics (1). Equation (2) then represents the ∞^2 conics each of which has *at least* three-point contact with $\rho = 0$ at P.

Since the general conic (2) depends on two parameters, we can subject it to two further conditions. Thus we are led to predict the following proposition.

THEOREM 4. *There is a unique point conic which has at least three-point contact with a given nondegenerate conic $\rho = 0$ at a given point P and passes through two other points P_1 and P_2, not both of which lie on the tangent t to $\rho = 0$ at P.*

Two conics satisfying the requirements would intersect in at least five points and hence be identical.*

* The alternative is that they both be degenerate and have in common the line t, and that t contain all the points of intersection. This is impossible because of the assumption that P_1 and P_2 do not both lie on t.

In proving that there is a conic satisfying the given conditions, we assume that P_1 is not on t and choose a point A on $\rho = 0$ not lying on PP_1. In the pencil of conics which have three-point contact with $\rho = 0$ at P and meet $\rho = 0$ again in A, there is a conic $\sigma = 0$ passing through P_1, and this conic, since P_1 lies on neither t nor PA, is nondegenerate. Again, in the pencil of conics which have three-point contact with $\sigma = 0$ at P and meet $\sigma = 0$ again in P_1, there is a conic $\kappa = 0$ which goes through P_2. This conic $\kappa = 0$ evidently passes through P_1 and P_2. Since it has three-point contact with $\sigma = 0$ at P and $\sigma = 0$ has like contact at P with $\rho = 0$, it has, by § 2, Th. 4,* at least three-point contact with $\rho = 0$ at P. It is therefore the conic sought.

COROLLARY. *There is a unique circle which has at least three-point contact with* $\rho = 0$ *at a finite point* P.

The corollary implies that this circle, of all the circles tangent to $\rho = 0$ at P, has the highest contact. Accordingly, it is the *osculating circle*, and its center and radius are the center of curvature and radius of curvature of the conic $\rho = 0$ at the point P.

The conic determined by the conditions of Theorem 4 has, in general, precisely three-point contact with the given conic at the given point. That the contact may, however, be four-point contact is clear from the proof of the theorem. An example in which this is actually the case is to be had in the osculating circle to a conic at a vertex.

Two-Point Contact. Corresponding to the fact that a pencil of Type II contains two distinct degenerate conics, there are two choices here of the "most general" degenerate conic having two-point contact with $\rho = 0$ at P. We may take it as $\alpha\tau = 0$, where $\tau = 0$ is, as before, the tangent to $\rho = 0$ at P and $\alpha = 0$ is now any line not passing through P; or as $\beta\gamma = 0$, where $\beta = 0$, $\gamma = 0$ are any lines through P distinct from $\tau = 0$. We thus obtain for the conics having two-point contact with $\rho = 0$ at P the alternative equations

$$(3) \qquad m\,\rho + \alpha\tau = 0, \qquad m\,\rho + \beta\gamma = 0.$$

THEOREM 5. *There are* ∞^3 *point conics having three-point contact with* $\rho = 0$ *at* P.

If we remove the restriction on $\alpha = 0$ and permit either or both

* The theorem applies only when $\kappa = 0$, as well as $\rho = 0$ and $\sigma = 0$, are nondegenerate. But if $\kappa = 0$ is degenerate, it consists of t and PP_1 and obviously has three-point contact with $\rho = 0$ at P.

of the lines $\beta = 0$, $\gamma = 0$ to coincide with $\tau = 0$, each of the equations (3) represents the ∞^3 conics which have *at least* two-point contact with $\rho = 0$ at P.

Double Contact. The point conics which have double contact with $\rho = 0$ at two points P_1, P_2 form with $\rho = 0$ a pencil of Type III. Alternative equations for them are

$$(4) \qquad m\,\rho + \tau_1\tau_2 = 0, \qquad m\,\rho + \gamma^2 = 0,$$

where $\tau_1 = 0$, $\tau_2 = 0$ are the tangents to $\rho = 0$ at P_1, P_2 and $\gamma = 0$ is the line P_1P_2.

EXERCISES

Find the equations of the following conics.

1. The conic which has four-point contact with $x^2 + 2\,y^2 = 3$ at the point (1, 1) and goes through the origin.

2. The conic which has double contact with $x^2 + 4\,y^2 = 100$ at the points (6, 4) and (6, $-$ 4) and contains the point (3, 4).

3. The conics which have at least three-point contact with the parabola $y^2 - 2\,x + y - 4 = 0$ at the point (1, 2). In particular, that one of these conics which passes through the points ($-$ 1, 0) and (0, 0).

4. The osculating circle to $y^2 = 4\,x$ at the point (1, 2). Find the center and radius of curvature.

5. The conic which has at least two-point contact with the parabola of Ex. 3 at the point (4, 3), is tangent to the line $x + y + 2 = 0$ at the point ($-$ 1, $-$ 1), and goes through the point (0, $-$ 1).

6. Show, by reference to the text, that the demand upon a conic that it have at least k-point contact with a given conic at a given point places k conditions on the conic.

7. Projective Applications. The pairs of lines 1, 1, 2, 2, 3, 3 in Fig. 8 are known as pairs of opposite common chords of the two conics shown in the figure. In general, two lines which form a degenerate conic of the pencil of point conics determined by two given conics constitute *a pair of opposite common chords* * of these conics.

THEOREM 1. *If each of two point conics has double contact with a third, their chords of contact with the third and a pair of their opposite common chords are concurrent.*

Let $\rho = 0$ be the equation of the third conic and $\alpha = 0$ and $\beta = 0$

* The terminology is somewhat unfortunate in that pairs of opposite common chords, as defined, may include lines which are actually common tangents, for example, t_1, P_2P_3 in Fig. 9, or t_1, t_2 in Fig. 10.

the equations of the chords of contact of the first and second conics with the third. The first conic, since it is a member of the pencil determined by the conics $\rho = 0$ and $\alpha^2 = 0$, may have its equation written in the form

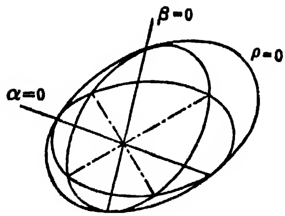


FIG. 13

(A) $\sigma \equiv \rho + k\,\alpha^2 = 0, \qquad k \neq 0.$

Similarly, the equation

(B) $\tau \equiv \rho + l\,\beta^2 = 0, \qquad l \neq 0,$

for a proper choice of l, represents the second conic.

A pair of opposite common chords of (A) and (B) constitutes a degenerate conic of the pencil determined by (A) and (B), and must be a linear combination of $\sigma = 0$ and $\tau = 0$. An obvious linear combination of $\sigma = 0$ and $\tau = 0$ which represents a degenerate conic is

$$\sigma - \tau \equiv k\,\alpha^2 - l\,\beta^2 = 0.$$

Hence the four lines in question are

$$\alpha = 0, \qquad \beta = 0, \qquad \sqrt{k}\,\alpha + \sqrt{l}\,\beta = 0, \qquad \sqrt{k}\,\alpha - \sqrt{l}\,\beta = 0.$$

They not only are concurrent, but form a harmonic set.*

THEOREM 2. *If three point conics have a common chord, and the three conics are taken in pairs and the common chord of each pair which is opposite to the given common chord is drawn, the three resulting lines are concurrent.*

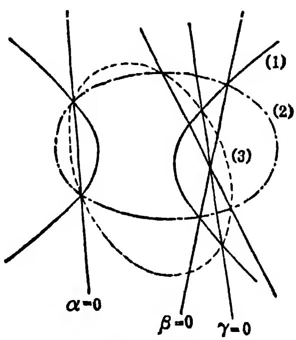


FIG. 14

Let the equation of the conic (1) in Fig. 14 be

(1) $\rho = 0,$

and let that of the common chord of the three conics be $\alpha = 0$.

If $\beta = 0$ is the common chord of conics

* *Critique.* If $\rho = 0$ is a doubly counting line, the theorem is not in general true. If $\alpha = 0$ and $\beta = 0$ are identical or if $\sigma = 0$ or $\tau = 0$ is a doubly counting line, the theorem is trivial. When these possibilities are excluded, the proof is impeccable. The conics $\rho = 0$ and $\alpha^2 = 0$ *cannot* be identical; $\sigma = 0$ *is* therefore a linear combination of them and, since $\sigma = 0$ cannot be $\alpha^2 = 0$, this linear combination *can* be written in the form (A). Equation (B) is similarly justified. Finally, inasmuch as $kl \neq 0$ and $\alpha = 0$, $\beta = 0$ are distinct, the four lines in question *do* form a harmonic set.

(1) and (2) which is opposite to $\alpha = 0$, then $\alpha\beta = 0$ is a degenerate conic of the pencil determined by the conics (1) and (2), and the conic (2) has an equation of the form

(2) $$\sigma \equiv k\,\rho + \alpha\beta = 0.$$

Similarly, the equation

(3) $$\tau \equiv l\,\rho + \alpha\gamma = 0,$$

where $\gamma = 0$ is the common chord of (1) and (3) which is opposite to $\alpha = 0$, represents, for a proper choice of l, the conic (3).

The common chord of (2) and (3) which is opposite to $\alpha = 0$ constitutes with $\alpha = 0$ a degenerate conic of the pencil determined by $\sigma = 0$ and $\tau = 0$. Since this degenerate conic evidently is

$$l\,\sigma - k\,\tau \equiv \alpha(l\,\beta - k\,\gamma) = 0,$$

the desired common chord of (2) and (3) is $l\,\beta - k\,\gamma = 0$.

We now have as the equations of the required lines $\beta = 0$, $\gamma = 0$, $l\,\beta - k\,\gamma = 0$, and the theorem is established.*

The theorem yields Pascal's Theorem as a special case (Fig. 15). The given conic Q and the degenerate conics 1, 5 and 2, 4 have the line BE as a common chord. The opposite common chord of Q and 1, 5 is 6; that of Q and 2, 4 is 3; and that of 1, 5 and 2, 4 is the line which joins the point of intersection of 1 and 4 with that of 2 and 5. Hence the point of intersection of 3 and 6 lies on this line.

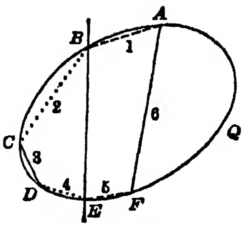


FIG. 15

THEOREM 3. *If each of three point conics has double contact with a fourth point conic, and the three conics are taken by twos and a certain pair of opposite common chords are drawn, the six resulting lines pass by threes through four points.*

Brianchon's Theorem may be obtained as a special case of this theorem.

* *Critique.* If two conics are degenerate and have a line in common, they have, not one, but infinitely many common chords opposite to this line. Accordingly, we assume that at most one of the given conics contains $\alpha = 0$ and that $\rho = 0$, in particular, does not contain $\alpha = 0$. Since $\rho = 0$ is then distinct from $\alpha\beta = 0$ and $\alpha\gamma = 0$, the conics (2) and (3) can have their equations written in the forms (2) and (3) and, inasmuch as at most one of these conics contains $\alpha = 0$, at most one of the constants k, l can vanish. Thus the proof of the theorem is checked.

EXERCISES

1. What does Theorem 1 become when $\sigma = 0$ and $\tau = 0$ are taken as degenerate conics, consisting of pairs of tangents to the nondegenerate conic $\rho = 0$ and not having a line in common?

Ans. A special case of Brianchon's Theorem.

2. The same, if $\rho = 0$ is a degenerate conic consisting of two distinct lines and $\sigma = 0$ and $\tau = 0$ are nondegenerate.

3. Prove that two opposite common chords of a hyperbola and an ellipse which is tangent to the asymptotes of the hyperbola are parallel to the chord of contact of the ellipse with these asymptotes and equally distant from it.

4. What does Theorem 2 become in each of the following cases?

(*a*) When the two points common to the three conics are I and J.

(*b*) When just two of the conics are circles.

(*c*) When the three conics, instead of having a common chord, are all tangent to a given line at a given point.

(*d*) When each of the conics consists of two distinct lines passing respectively through two given points.

5. Two nondegenerate conics intersect in two real, and two conjugate-imaginary, points. Give a construction for the common chord determined by the two imaginary points.

6. Four points A, B, C, D trace respectively the four sides of a parallelogram so that the lines AC and BD remain always parallel to sides of the parallelogram. Find the locus of the point of intersection of the lines AB and CD.

7. Prove Theorem 3 and criticize both the theorem and the proof.

8. Obtain Brianchon's Theorem as a special case of Theorem 3.

9. If through the point of contact of two nondegenerate conics of Type II a line be drawn meeting the conics again in P_1 and P_2, the tangents to the conics at P_1 and P_2 intersect on the common chord of the two conics which is opposite to their common tangent.

8. Similar and Similarly Placed Conics.

DEFINITION. *Two nondegenerate conics are said to be similar and similarly placed when they have the same eccentricity * and their corresponding axes are parallel.*

Central Conics. If two hyperbolas are similar and similarly placed, the asymptotes of the one are parallel to those of the other. This is true also of two conjugate hyperbolas or, more generally, of any two hyper-

* In the case of a central conic, which has two eccentricities (Ch. VIII, § 9), we mean here the eccentricity associated with the axis containing the real foci. This is the usual eccentricity of elementary analytic geometry when the conic has a real trace. For the representative conic without a real trace, $x^2/a^2 + y^2/b^2 = -1$, $a > b$, it is c/bi; the real foci of this conic lie on the y-axis.

bolas each of which is similar and similarly placed to the conjugate of the other. In either case, the two hyperbolas intersect the line at infinity in the same points. Conversely, if two hyperbolas have the same points at infinity, they are similar and similarly placed or each is similar and similarly placed to the conjugate of the other.

These considerations apply equally well to ellipses provided we define two ellipses as conjugate whose equations can be reduced to the forms

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \qquad \frac{x^2}{a^2} + \frac{y^2}{b^2} = -1$$

by the same change of metric coordinates. Hence:

THEOREM 1. *Two central conics are similar and similarly placed or each is similar and similarly placed to the conjugate of the other if and only if they have the same points at infinity.*

The following theorem is now readily proved.

THEOREM 2. *There are ∞^3 central conics which are similar and similarly placed to a given central conic $\rho = 0$ or to its conjugate. They are the nondegenerate conics defined by the equation*

$$\rho + k\,\alpha x_3 = 0, \tag{1}$$

where $\alpha = 0$ is an arbitrary line.

Circles have been tacitly excluded from the foregoing discussion inasmuch as the definition is inapplicable to them; see Ex. 4.

Parabolas. Two parabolas are always similar. They are similarly placed if their axes are parallel, that is, if they are tangent to the line at infinity at the same point.

THEOREM 3. *There are ∞^3 parabolas similarly placed to a given parabola $\rho = 0$. They are the nondegenerate conics defined by the equation*

$$\rho + k\,\alpha x_3 = 0, \tag{2}$$

where $\alpha = 0$ is an arbitrary line.

From equations (1) and (2) we conclude that, if the coefficients of the quadratic terms in the equations in nonhomogeneous coördinates of two nondegenerate conics are equal or proportional, the two conics are similar and similarly placed or each is similar and similarly placed to the conjugate of the other, and conversely.

EXERCISES

1. Prove that two parabolas have four-point contact at infinity if and only if they are congruent, have the same axis, and open in the same direction.

2. There are ∞^2 parabolas with a given line as axis. Derive an equation which represents them all.

3. Under what conditions will two parabolas have three-point contact at infinity?

4. Show that, if two circles are said to be similar and similarly placed if they both have real traces or both are without real traces, and a definition of conjugate circles analogous to that of conjugate ellipses is introduced, Theorem 2 is valid when $\rho = 0$ is a circle.

5. Prove that *two nondegenerate conics are similar and similarly placed if they are equivalent* (Ch. XIV, § 4) *with respect to the group of homothetic transformations* *

$$x' = \rho x + a, \qquad y' = \rho y + b, \qquad \rho \neq 0.$$

6. Prove that, if a conic has double contact in finite points with one of two similar, similarly placed, and concentric central conics, two opposite chords which it has in common with the other are parallel to the chord of contact and equally distant from it.

7. Discuss in detail the central conics which are concentric with a given central conic and are similar and similarly placed either to it or to its conjugate.

8. Show that, if for each two of three similar and similarly placed conics a certain finite common chord is drawn, the three resulting lines are concurrent.

9. It is evident geometrically that a pencil of similar and similarly placed conics contains no parabolas or consists exclusively of parabolas. Verify analytically and show that this is the only case in which a pencil of point conics fails to contain just two parabolas, distinct or coincident.

9. The Biquadratic Equation. The equation

$$(1) \qquad f(t) \equiv t^4 + a_1t^3 + a_2t^2 + a_3t + a_4 = 0$$

is usually solved by factoring the polynomial $f(t)$ into two quadratic factors. The algebraic devices by means of which the factorization is effected are not easily kept in mind. In their stead we shall introduce a geometric device, whereby $f(t)$ is replaced by a *quadratic* polynomial $F(x, y)$ whose discriminant vanishes. The problem of factoring $f(t)$ into quadratic factors will then be reduced to the problem of factoring $F(x, y)$ into linear factors, that is, to the familiar problem of finding the constituent lines of the degenerate point conic $F(x, y) = 0$.

The polynomial $F(x, y)$ is obtained from $f(t)$ by replacing the powers of t in $f(t)$ by properly chosen products of powers of x and y. Each of

* This group of homothetic transformations is more general than that of Ex. 2, End of Ch. VII, in that here ρ is not restricted to be positive. The content of the theorem might have been taken as definition, in the place of the separate definitions for circles and nondegenerate conics, not circles; it is also applicable to degenerate conics.

the powers of t, except t^2, can be written in just one way as the product of at most two factors of the forms t^2 and t, as follows: $t^4 = t^2 \cdot t^2$, $t^3 = t^2 \cdot t$, $t = t$. On the other hand, t^2 can be expressed either as t^2 or as $t \cdot t$. If, then, we write

$$f(t) \equiv t^2 \cdot t^2 + a_1 t^2 \cdot t + \lambda\, t \cdot t + (a_2 - \lambda)t^2 + a_3 t + a_4,$$

and replace t^2 and t respectively by x and y, we obtain the desired polynomial:

$$(2) \qquad F(x, y) \equiv x^2 + a_1 xy + \lambda\, y^2 + (a_2 - \lambda)x + a_3 y + a_4.$$

If the parameter λ is so chosen that the conic $F(x, y) = 0$ is degenerate, $F(x, y)$ can be factored into linear factors:

$$F(x, y) \equiv (x + b_1 y + c_1)(x + b_2 y + c_2).$$

Hence

$$F(t^2, t) \equiv (t^2 + b_1 t + c_1)(t^2 + b_2 t + c_2).$$

But it is evident from (2) that $F(t^2, t) \equiv f(t)$ and therefore

$$f(t) \equiv (t^2 + b_1 t + c_1)(t^2 + b_2 t + c_2).$$

Thus we have established the following theorem.

THEOREM 1. *If λ is so chosen that $F(x, y)$ is factorable into linear factors, these factors become the quadratic factors of $f(t)$ when for x and y are set t^2 and t respectively.*

The values of λ for which $F(x, y)$ is factorable are the solutions of the equation obtained by equating the discriminant of $F(x, y)$ to zero:

$$(3) \quad \lambda^3 - 2\, a_2\lambda^2 + (a_1 a_3 + a_2^2 - 4\, a_4)\lambda + (a_1^2 a_4 - a_1 a_2 a_3 + a_3^2) = 0.$$

This equation is known as the *cubic resolvent* of the given biquadratic equation.

Example. Let the given equation be

$$f(t) = t^4 + t^3 - 7\, t^2 + 2\, t + 4 = 0.$$

Then

$$F(x, y) \equiv x^2 + xy + \lambda\, y^2 - (7 + \lambda)x + 2\, y + 4,$$

and (3) becomes

$$\lambda^3 + 14\, \lambda^2 + 35\, \lambda + 22 = 0.$$

This equation has as its roots -1, -2, -11. Of these three values of λ, only the second, $\lambda = -2$, renders the quadratic terms in $F(x, y)$ factorable into linear factors with *rational* coefficients. Setting

$\lambda = -2$ in $F(x, y)$ and factoring, we have

$$F(x, y) \equiv (x - y - 1)(x + 2y - 4).$$

Hence

$$f(t) \equiv (t^2 - t - 1)(t^2 + 2t - 4),$$

and the roots of $f(t) = 0$ are

$$\tfrac{1}{2}(1 + \sqrt{5}), \qquad \tfrac{1}{2}(1 - \sqrt{5}), \qquad -1 - \sqrt{5}, \qquad -1 + \sqrt{5}.$$

Geometric Interpretation. The equation

$$F(x, y) \equiv (x^2 + a_1xy + a_2x + a_3y + a_4) + \lambda(y^2 - x) = 0,$$

where λ is arbitrary, represents the pencil of point conics whose base conics are

$$(4) \qquad x^2 + a_1xy + a_2x + a_3y + a_4 = 0, \qquad y^2 - x = 0.$$

The roots of the cubic resolvent (3) determine the degenerate conics of the pencil.

The equation which results from the elimination of x from equations (4) is evidently $f(y) = 0$. Hence:

THEOREM 2. *The solutions of the biquadratic equation are the y-coordinates of the points common to all the conics of the pencil.*

Corresponding to the five types of pencils of conics are the five possibilities as to the multiplicities of the roots of the biquadratic equation. Thus, if the pencil of conics is of Type IV, the biquadratic equation has one triple root and one simple root.*

If the pencil of conics is of Type I, the roots of the biquadratic equation are distinct. The polynomial $f(t)$ can then be factored into quadratic factors in three essentially different ways, which correspond respectively to the three distinct degenerate conics of the pencil and to the three corresponding solutions of the cubic equation (3).

Graphical Solution. By the usual method † of adding a suitable quantity to the roots of (1), a_1 can be made zero. We assume that thereby a_3 is not also made zero; otherwise $f(t)$ would be quadratic in t^2.

* It is essential in this connection to note that, inasmuch as one of the base conics is the parabola $y^2 - x = 0$, two distinct points common to the conics of the pencil can never have the same y-coordinates. Otherwise a double root, let us say, of the biquadratic equation might correspond to two distinct points of intersection of the conics of the pencil and the correspondence between the five possibilities for the roots of the equation and the five types of pencils would be vitiated.

† See, for example, Rietz and Crathorne, *College Algebra*, Revised Ed., § 98.

Since $a_3 \neq 0$, the roots can now be multiplied by a quantity so chosen as to make $a_3 = -1$. Thus, $f(t)$ is reduced to the form

$$f(t) \equiv t^4 + 2\,b_2t^2 - t + b_4. \tag{5}$$

The base conics of the pencil are now the two congruent parabolas

$$[x + b_2]^2 - [y - (b_4 - b_2^2)] = 0, \qquad y^2 - x = 0,$$

and the y-coordinates of the points of intersection of these two parabolas are the roots of the reduced equation (5).

The following method of finding graphically the real roots immediately presents itself. On a sheet of coordinate paper construct the parabola $y^2 - x = 0$. On a sheet of tracing paper draw, to the same scale, the parabola $x'^2 - y' = 0$. Place the tracing paper on the coordinate paper so that the axes of the two parabolas are perpendicular and the vertex of the parabola on the tracing paper covers the point $(-b_2, b_4 - b_2^2)$ on the coordinate paper, and then read off the ordinates of the points of intersection of the two parabolas.

EXERCISES

Solve each of the following equations algebraically and graphically.

1. $t^4 + t^3 - 6\,t^2 - 5\,t + 3 = 0.$
2. $t^4 + t^3 - 11\,t^2 - 10\,t + 4 = 0.$
3. $t^4 + 6\,t^3 + 5\,t^2 - 12\,t + 4 = 0.$
4. $4\,t^4 - 7\,t^2 - 5\,t - 1 = 0.$

B. Line Conics

10. Pairs of Line Conics. We content ourselves in this and the following paragraph with a statement of the facts, leaving the proofs largely to the reader.

Theorem 1. *Two distinct line conics which have not a point in common always have four common tangents.*

In the general case in which the given line conics Q_1, Q_2 are nondegenerate, the theorem may be established by dualizing the proof of Theorem 1, § 2. The basic lines of a system of projective line coordinates are chosen as follows: a_2, a_3, d as tangents to Q_1, a_2 not also tangent to Q_2; and a_1 as the chord of contact of the tangents a_2, a_3. The equations of Q_1, Q_2 in the forms which they then assume in the nonhomogeneous coordinates $u = u_1/u_3$, $v = u_2/u_3$, namely

$$v = u^2 \tag{1}$$

$$au^2 + buv + cv^2 + du + ev + f = 0, \qquad c \neq 0, \tag{2}$$

have the four simultaneous solutions (r_i, r_i^2), $i = 1, 2, 3, 4$, where r_1, r_2, r_3, r_4 are the roots of

$$cu^4 + bu^3 + (a + e)u^2 + du + f = 0, \qquad c \neq 0.$$

Consequently, Q_1 and Q_2 have in common the four tangents

$$T_i: \qquad (r_i, r_i^2, 1), \qquad (i = 1, 2, 3, 4).$$

EXERCISE. Expand the foregoing proof, justifying the assumption that $u_3 \neq 0$ and interpreting geometrically the method by which equations (1) and (2) are solved.

THEOREM 2. *The two line conics are never tangent to one another on a simple common tangent and are always tangent to one another on a multiple common tangent.*

This statement says, in the case in which the conics are nondegenerate, that the points of contact of Q_1 and Q_2 with a common tangent T coincide when and only when T counts at least twice as a common tangent. When one or both of the conics are degenerate, it is no longer to be taken as a proposition to be proved, but as a definition of tangency of Q_1 and Q_2.

DEFINITION. *If two line conics are tangent to one another on a common tangent T, they are said to have two-line, three-line, or four-line contact according as T is a double, triple, or quadruple common tangent. If the two conics have two-line contact on each of two lines, they are said to have double contact on these lines.*

We are now in a position to classify pairs of line conics according to the nature of their common tangents.

I: $[T_1, T_2, T_3, T_4]$. Q_1 and Q_2 have four distinct common tangents. Each of these is a simple common tangent: Q_1 and Q_2 are nowhere tangent to one another.

II: $[T_1^2, T_2, T_3]$. Q_1 and Q_2 have two-line contact on T_1 and have T_2 and T_3 as simple common tangents.

III: $[T_1^2, T_2^2]$. Q_1 and Q_2 have double contact on the lines T_1 and T_2.

IV: $[T_1^3, T_2]$. Q_1 and Q_2 have three-line contact on T_1 and have T_2 as a simple common tangent.

V: $[T_1^4]$. Q_1 and Q_2 have four-line contact on T_1.

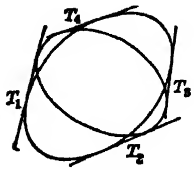


FIG. 16

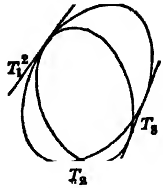


FIG. 17

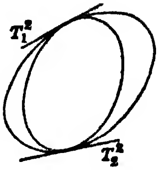


FIG. 18

The five types are illustrated, for nondegenerate conics, by Figs. 16–20. In Fig. 21 we have a special case of Type II in which Q_2 consists of two distinct points lying on a tangent to Q_1, and in Fig. 22, a case of double contact in which both Q_1 and Q_2 are degenerate, one of them consisting of a single point counted twice.

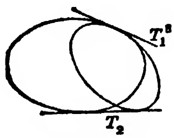


FIG. 19

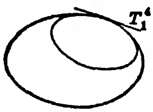


FIG. 20

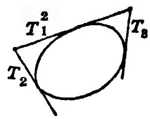


FIG. 21

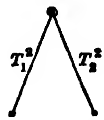


FIG. 22

By means of the method employed in the proof of Theorem 1, the following proposition can be established.

THEOREM 3. *If two nondegenerate conics have k-line contact with a given nondegenerate conic on a line T, they have at least k-line contact with each other on T.*

EXERCISES

1. Discuss pairs of line conics from a geometrical standpoint analogous to that of § 1.

2. Prove Theorem 2, in conjunction with the Exercise in the text.

3. Construct an example of a pair of nondegenerate conics of Type IV. Then prove Theorem 3 when $k = 3$.

4. Show that, if two nondegenerate conics have double contact on two lines, they have double contact at two points, and vice versa.

5. Discuss in detail the tangents common to a nondegenerate and a degenerate line conic.

6. Exhibit a pair of degenerate line conics of Type II.

11. Pencils of Line Conics. The totality of line conics

$$k\rho + l\sigma = 0, \tag{1}$$

linearly dependent on two distinct line conics,

$$\rho = \sum a_{ij}u_iu_j = 0, \qquad \sigma = \sum b_{ij}u_iu_j = 0,$$

is called a pencil of line conics.

A pencil of line conics has the following properties.

A. Any two distinct conics of the pencil may be taken as the base conics.

B. There is a unique conic of the pencil tangent to a given line of the plane other than the tangents common to all the conics of the pencil.

C. The pencil contains three degenerate line conics.*

D. The tangents common to two conics of the pencil and the number of times each counts are the same for each two conics of the pencil.

According to Property D there are five types of pencils of line conics, corresponding to the five types of pairs of line conics.

I: $[T_1, T_2, T_3, T_4]$. *The pencil consists of all the line conics tangent to four lines, no three of which are concurrent.*

The degenerate conics of the pencil consist of the pairs of opposite vertices of the complete quadrilateral formed by the four lines. Taking two of these conics, for example $\alpha\beta = 0$ and $\gamma\delta = 0$, where $\alpha = 0$, $\beta = 0$, $\gamma = 0$, $\delta = 0$ are the equations in line coordinates of the points indicated in the figure, as new base conics, we obtain

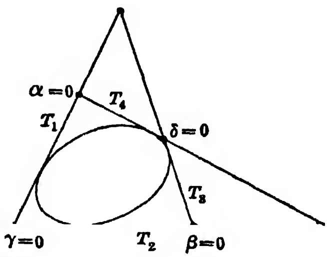


FIG. 23

(2) $$k\,\alpha\beta + l\,\gamma\delta = 0$$

as a simple equation of the pencil.

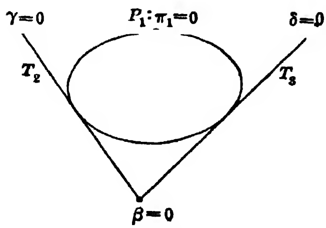


FIG. 24

Let the reader show how these results may be applied to find the equation of the line conic which is tangent to five given lines.

II: $[T_1^2, T_2, T_3]$. *The pencil consists of all the line conics which are tangent at a given point P_1 to a given line T_1 and tangent to two other lines T_2, T_3.*

There are two distinct degenerate conics, $\pi_1\beta = 0$ and $\gamma\delta = 0$ (Fig. 24). Which one counts twice?

* We exclude here the cases in which all the conics of the pencil are degenerate.

Taking the degenerate conics as base conics, we obtain a new representation of the pencil,

$$k\,\pi_1\beta + l\,\gamma\delta = 0, \tag{3}$$

by means of which we can readily solve the problem of finding the equation of the line conic tangent to a given line at a given point, and to three other lines.

III: $[T_1^2,\ T_2^2]$. *The pencil consists of all the line conics which are tangent at two given points P_1 and P_2 to given lines T_1 and T_2.*

The line conics $\pi_1\pi_2 = 0$ and $\gamma^2 = 0$, Fig. 25, are the degenerate conics of the pencil and the equation of the pencil based on them is

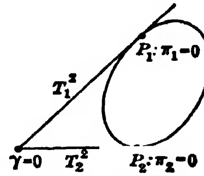


FIG. 25

$$k\,\pi_1\pi_2 + l\,\gamma^2 = 0. \tag{4}$$

What general problem can be solved by the aid of this equation?

IV: $[T_1^3,\ T_2]$. *The pencil consists of all the line conics which have three-line contact with a given nondegenerate conic on a given line T_1 and have in common with this conic a second tangent T_2.*

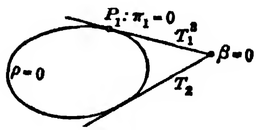


FIG. 26

The pencil contains just one (triply counting) degenerate conic, $\pi_1\beta = 0$. If $\rho = 0$ is one of its nondegenerate conics, a new equation for it is

$$k\,\rho + l\,\pi_1\beta = 0. \tag{5}$$

V: $[T_1^4]$. We leave to the reader the discussion of this case.

EXERCISES

Find the equations in line coordinates of the following conics.

1. The conic tangent to the five lines (1, 0, 1), (− 1, 0, 1), (0, 1, 1), (0, − 1, 1), (1, 1, 1).

2. The parabola in the pencil of line conics determined by $2\,uv = 3$, $u^2 + v^2 - 1 = 0$.

3. The hyperbola which has $x - y = 0$ as an asymptote and is tangent to the lines $x + y = 0$, $x = 3$, $5\,x - 4\,y = 4$.

4. The conic which is tangent to $x_1 + x_2 - x_3 = 0$ at (1, 0, 1), to $x_1 - x_2 = 0$ at (0, 0, 1), and to $x_1 - 4\,x_3 = 0$.

5. The parabola which has an axis of slope unity and is tangent to the line (− 1, − 1) and to the coordinate axes.

6. The hyperbolas of § 4, Ex. 5.

7. Show that the pencils of point conics of Type III and the pencils of line conics of Type III are identical, except for their degenerate conics.

8. Discuss a pencil of line conics of Type V. Then prove that the pencils of point conics of Type V and the pencils of line conics of Type V are identical except for their degenerate conics, provided it is assumed that four-point contact and four-line contact of two nondegenerate conics are equivalent.

9. How many parabolas does a pencil of line conics contain?

10. The pencils of point conics of Type III and the pencils of line conics of Type III are essentially the same. But, in general, a pencil of point conics contains two parabolas and a pencil of line conics only one. Explain the paradox.

12. Applications to Foci. Confocal Conics. We recall that the tangents to a nondegenerate conic from a focus are isotropic lines and, conversely, that two isotropic tangents which are not parallel intersect in a focus.

THEOREM 1. *A necessary and sufficient condition that a nondegenerate conic have a given point as a focus is that it be tangent to the isotropics through the point.*

Thus, the demand that a conic have a given point F as a focus places on it two independent conditions. Hence, *the general conic with F as a focus depends on three parameters.* It may, then, be subjected to three further conditions. We may, for example, demand that it be tangent to three prescribed lines.

Confocal Conics. Two foci of a nondegenerate central conic, not a circle, which lie on the same axis of the conic we shall call a *pair of foci.* There are two pairs of foci, one on each axis. The lines joining the foci of the one pair to those of the other are the isotropic tangents to the conic.

The isotropic tangents may also be thought of as the isotropic lines which pass through the foci on *one* axis. The foci on the other axis are, then, the other two finite points of intersection of these isotropic lines. Hence the two pairs of foci cannot be prescribed independently. When one pair is given, the other is determined.

THEOREM 2. *Two nondegenerate central conics which have one pair of foci in common have all their foci in common.*

It is clear from our argument that two pairs of finite points can serve as the foci of a central conic if and only if the four isotropic lines issuing from the points of one pair intersect also in the points of the other pair. The two pairs of points are then the foci of every nondegenerate conic which is tangent to the four isotropic lines.

The totality of these conics consists of the nondegenerate conics of a pencil of line conics of Type I. The equation of the pencil, if $\alpha = 0$, $\beta = 0$ and $\gamma = 0$, $\delta = 0$ represent the two pairs of points, is

(1) $$k\,\alpha\beta + l\,\gamma\delta = 0.$$

The totality is usually known as a set of confocal central conics. It consists of all the central conics having one pair, and hence all four, of the given points as foci.

THEOREM 3. *A set of confocal central conics consists of the nondegenerate conics of a pencil of line conics which is made up of the conics tangent to four isotropic lines, two of each kind.*

As an illustration, we shall find the equation of the central conics which have $(\pm c, 0)$ as one pair of foci and hence $(0, \pm ic)$ as the second pair. Here

$$\alpha \equiv cu + 1 = 0, \qquad \gamma \equiv icv + 1 = 0,$$
$$\beta \equiv cu - 1 = 0, \qquad \delta \equiv -icv + 1 = 0.$$

Hence the desired equation is

(2) $$k\,(c^2u^2 - 1) + l\,(c^2v^2 + 1) = 0, \qquad k\,l(k - l) \neq 0.^*$$

To find the equation in point coordinates we rewrite (2) in the form

$$\frac{ku^2}{k - l} + \frac{lv^2}{k - l} = \frac{1}{c^2}.$$

Hence we have

$$\frac{x^2}{\dfrac{k}{k - l}} + \frac{y^2}{\dfrac{l}{k - l}} = c^2.$$

Noticing that

$$\frac{2\,k}{k - l} - \frac{2\,l}{k - l} = 2,$$

we are led to set

$$\frac{2\,k}{k - l} = \lambda + 1 \qquad \text{whence} \qquad \frac{2\,l}{k - l} = \lambda - 1.$$

We thus obtain as the equation in point coordinates

$$\frac{x^2}{\lambda + 1} + \frac{y^2}{\lambda - 1} = \frac{c^2}{2}, \qquad \lambda^2 \neq 1.$$

For values of λ between -1 and $+1$, the equation represents hyper-

* The degenerate conics of the pencil are excluded.

bolas; for $\lambda > 1$, ellipses with real traces; and for $\lambda < -1$, ellipses without real traces.

Two nondegenerate parabolas are said to be confocal if they have the same focus and the same axis. This is equivalent to requiring that they have the same isotropic tangents and the same point at infinity; for, the isotropic tangents of a parabola determine the focus, and the focus and the point at infinity on the parabola then determine the axis.

THEOREM 4. *A set of confocal parabolas is essentially a pencil of line conics of Type* II. *It consists of the nondegenerate parabolas which are tangent to the isotropic lines through the common focus and tangent to the line at infinity at the point at infinity in the direction of the common axis.*

EXERCISES

1. Find the equation of the parabola which has the point (0, 1) as focus and is tangent to the lines $2x - y - 2 = 0$, $2x - 2y - 1 = 0$.

2. Show that the conics which have a given point F as a focus and a given line D, not passing through F, as the corresponding directrix form a pencil.

3. Find the equation of the central conic which has the points $(-1, 2)$ and $(2, -1)$ as foci and is tangent to the axis of x.

4. The same problem, if the conic is to go through the origin instead of being tangent to the axis of x. There are two answers. Why?

5. By means of Th. 4 find the equation in line coordinates of the confocal parabolas which have the origin as focus and the axis of x as axis. Show that the resulting equation in point coordinates may be put into the form

$$y^2 = 2\lambda x + \lambda^2, \qquad \lambda \neq 0.$$

6. Discuss the conics which have a given finite line as directrix.

13. Further Applications. Two points which form a degenerate conic of the pencil of line conics determined by two given line conics are called a *pair of opposite intersections of common tangents* of these conics.

THEOREM 1. *If three line conics have two tangents in common and the three conics are taken in pairs and the intersection of the common tangents of each pair which is opposite to the intersection of the given common tangents is marked, the three points obtained are collinear.*

This is the dual of Theorem 2 of § 7.

THEOREM 2. *The dual of Theorem* 1 *of* § 7.

EXERCISES

1. Prove Theorem 1 and show that Brianchon's Theorem is a special case of it.

2. Deduce from Theorem 1 a proposition concerning three line conics with a common focus.

3. Three line conics have two tangents t_1, t_2 in common and each two of them are mutually tangent, at points not lying on either t_1 or t_2. Show that the three points of contact are collinear.

4. State and prove Theorem 2.

5. What does Theorem 2 become (*a*) if the odd conic consists of two distinct points and the other two conics are nondegenerate? (*b*) If the conditions are reversed?

6. State and prove the theorem dual to Theorem 3 of § 7.

7. As a special case of the theorem of Ex. 6, show that the real common tangents of three mutually intersecting circles,* taken in pairs, form a Pascal hexagon.

C. Point Conics and Line Conics

14. The Involution Theorems of Desargues and Sturm.

Theorem 1 *a* (Desargues). *The pairs of points in which a line meets the conics of a pencil of point conics form an involution, provided the line does not contain a point common to all the conics of the pencil.*

Theorem 1 *b* (Sturm). *The pairs of tangents from a point to the conics of a pencil of line conics form an involution, provided the point does not lie on a tangent common to all the conics of the pencil.*

In proving the Theorem of Desargues, we choose our projective coordinate system so that the given line L is $x_3 = 0$. The pairs of points in which L cuts two distinct conics of the pencil,

$$\sum a_{ij}x_ix_j = 0, \qquad \sum b_{ij}x_ix_j = 0,$$

are defined, then, by the quadratic equations

$$a_{11}x_1^2 + 2\,a_{12}x_1x_2 + a_{22}x_2^2 = 0, \qquad b_{11}x_1^2 + 2\,b_{12}x_1x_2 + b_{22}x_2^2 = 0.$$

These two pairs of points have not a point in common, since L would otherwise contain a point common to all the conics of the pencil. Hence they determine an involution. According to Ex. 12, End of Ch. IX, the pairs of points in this involution are represented by the equation

$$(1) \quad k\,(a_{11}x_1^2 + 2\,a_{12}x_1x_2 + a_{22}x_2^2) + l\,(b_{11}x_1^2 + 2\,b_{12}x_1x_2 + b_{22}x_2^2) = 0,$$

* That is, three circles each two of which have *real* points of intersection.

and are therefore the pairs of points in which $L : x_3 = 0$ meets the conics

(2) $$k \sum a_{ij} x_i x_j + l \sum b_{ij} x_i x_j = 0$$

of the pencil.*

Let us now turn our attention to the Theorem of Sturm. Let P be a point not lying on a common tangent of the given pencil of line conics, and let d be a double line of the involution consisting of the pairs of tangents drawn from P to the conics of the pencil. There is, we know, a unique conic of the pencil which is tangent to d. The second tangent from P to this conic is the mate of d in the involution and therefore coincides with d. In other words, the two tangents from P to the conic are identical. Hence P is a point of the conic. Conversely, if a conic of the pencil passes through P, the tangents to it from P coincide and constitute one of the double lines of the involution. Hence:

THEOREM 2 *b*. *There are two distinct conics in a pencil of line conics which pass through a given point* † *not lying on a common tangent to the conics of the pencil. Their tangents at the point are the double lines of the involution established at the point by the pencil.*

* An elementary proof can be given along the same lines. Introduce on $L : x_3 = 0$ the nonhomogeneous coordinate $x = x_1/x_2$ and denote by ξ, ξ' the nonhomogeneous coordinates of the points in which L meets the arbitrary conic (2) of the pencil. Since ξ, ξ' are the roots of the quadratic equation

$$(k\,a_{11} + l\,b_{11})x^2 + 2\,(k\,a_{12} + l\,b_{12})x + (k\,a_{22} + l\,b_{22}) = 0,$$

we have

$$\frac{\xi + \xi'}{2} = -\frac{ka_{12} + lb_{12}}{ka_{11} + lb_{11}}, \qquad \xi\xi' = \frac{ka_{22} + lb_{22}}{ka_{11} + lb_{11}}.$$

In order to find how ξ is transformed into ξ', it is necessary to eliminate k, l. This is most easily done by rewriting the equations in the forms

$$ka_{11} + lb_{11} - \rho = 0, \quad ka_{12} + lb_{12} + \frac{\xi + \xi'}{2}\rho = 0, \quad k\,a_{22} + l\,b_{22} - \xi\xi'\rho = 0.$$

Hence

$$\begin{vmatrix} a_{11} & b_{11} & -1 \\ a_{12} & b_{12} & \dfrac{\xi + \xi'}{2} \\ a_{22} & b_{22} & -\xi\xi' \end{vmatrix} = 0.$$

The transformation of ξ into ξ' represented by this equation is evidently linear. It cannot be singular, by hypothesis, and consequently, since the equation is symmetric in ξ and ξ', it is an involution.

† It is evident from the proof that what we mean here by a "degenerate line conic passing through a point" is the same as what we always mean by a non-degenerate line conic passing through a point, namely that the lines of the conic through the point coincide.

As an application of the theorem we deduce the fact that a set of confocal conics (§ 12) forms an orthogonal system. Since the tangents at P to the two conics which pass through P are the double lines of the involution at P, they will be perpendicular if the isotropic lines through P are paired in the involution. But the circular points at infinity constitute a degenerate conic of the pencil and the tangents from P to this conic are the isotropics through P.

THEOREM 2 *a*. *There are two distinct conics in a pencil of point conics which are tangent to a given line* * *not passing through a point common to the conics of the pencil. Their points of contact with the given line are the double points of the involution in which the line cuts the pencil.*

As a direct corollary to this theorem we have

THEOREM 3 *a*. *Passing through four points, no three collinear, and tangent to a given line not containing any of the given points, there are two distinct conics.*

THEOREM 3 *b*. *The dual of Theorem* 3 *a*.

These theorems raise the question as to the number of conics passing through three points and tangent to two lines, or passing through two points and tangent to three lines. The answer is four. There are, for example, four circles tangent to three finite lines, namely, the inscribed circle and the three escribed circles of the triangle formed by the three lines.

If three points and two tangents are prescribed, the three points may be taken as the vertices of the triangle of reference. The general conic which contains them has the equation in point coordinates,

$$r_1x_2x_3 + r_2x_3x_1 + r_3x_1x_2 = 0,$$

and hence the equation in line coordinates,

$$r_1^2u_1^2 + r_2^2u_2^2 + r_3^2u_3^2 - 2\,r_1r_2u_1u_2 - 2\,r_2r_3u_2u_3 - 2\,r_3r_1u_3u_1 = 0.$$

This conic is tangent to the two lines (a_1, a_2, a_3), (b_1, b_1, b_3) if and only if

$$a_1^2r_1^2 + a_2^2r_2^2 + a_3^2r_3^2 - 2\,a_1a_2r_1r_2 - 2\,a_2a_3r_2r_3 - 2\,a_3a_1r_3r_1 = 0,$$
$$b_1^2r_1^2 + b_2^2r_2^2 + b_3^2r_3^2 - 2\,b_1b_2r_1r_2 - 2\,b_2b_3r_2r_3 - 2\,b_3b_1r_3r_1 = 0,$$

and these simultaneous equations, since they may be interpreted as representing two conics in (r_1, r_2, r_3) as point coordinates, furnish four solutions of our problem.

* What is meant here by a degenerate point conic being tangent to a line?

EXERCISES

1. Prove Theorem 2 *a*.

2. Prove that, if a pencil of point conics contains just two parabolas, the two parabolas are coincident if and only if the line at infinity contains a point common to all the conics of the pencil.

3. A set of hyperbolas all have the same asymptotes. Prove that the segments which they cut from a fixed line all have the same mid-point.

4. Show that, if the line at infinity is cut by a pencil of point conics in an involution which is elliptic, all the real conics of the pencil are hyperbolas. What are the facts if the involution is hyperbolic?

5. Prove that, if two nondegenerate conics have double contact, the tangent at a point P on one of them and the line joining P to the point of intersection of the common tangents are conjugate lines with respect to the other. State the dual theorem.

6. Show that the two conics of Theorem 2 *a* are both nondegenerate when and only when the given line does not contain a singular point of a degenerate conic of the pencil. Describe more specifically when the conics of Theorem 3 *a* are both nondegenerate.

7. Prove that one of the two conics of Theorem 2 *a* is always degenerate if the pencil is of Type III or V. What light does this throw on the fact that a pencil of either of these types is self-dual? See § 11, Exs. 7, 8.

15. Poles and Polars with respect to Pairs and Pencils of Conics. Let two nondegenerate conics be given by their equations in point coordinates,

$$\sum a_{ij}x_ix_j = 0, \qquad \sum b_{ij}x_ix_j = 0, \tag{1}$$

and let it be required to find the points r whose polars with respect to the two conics,

$$\sum a_{ij}r_ix_j = 0, \qquad \sum b_{ij}r_ix_j = 0, \tag{2}$$

are identical.

The lines (2) are the same if and only if a constant λ exists so that

$$\sum a_{ij}r_ix_j + \lambda \sum b_{ij}r_ix_j \equiv 0. \tag{3}$$

In order to interpret this identity geometrically we consider, in conjunction with it, the equation

$$\sum a_{ij}r_ix_j + \lambda \sum b_{ij}r_ix_j = 0, \tag{4}$$

formed for an arbitrary point r and an arbitrary value of λ. This equation represents the polar of the point r with respect to the arbitrary conic,

$$\sum a_{ij}x_ix_j + \lambda \sum b_{ij}x_ix_j = 0, \tag{5}$$

of the pencil of point conics determined by the given conics (1).

Since the polar of a point with respect to a point conic is undefined when and only when the point conic is degenerate and the point is one of its singular points (Ch. XIV, § 7), equation (4) becomes the identity (3) when and only when r is a singular point of a degenerate conic of the pencil (5) and λ is the constant value of the parameter in (5) which defines this degenerate conic.

THEOREM 1 *a*. *The singular points of the degenerate conics of the pencil of point conics determined by two nondegenerate conics have the same polars with respect to the two conics and are the only points with this property.*

When we think of the two given conics as line conics, a similar argument leads to the dual theorem.

THEOREM 1 *b*. *The only lines which have the same poles with respect to two nondegenerate conics are the singular lines of the degenerate conics of the pencil of line conics determined by the two conics.*

It is clear that the singular lines of the second theorem are the polars of the singular points of the first; and vice versa. For, if a point P has the same polar L with respect to both conics, the line L has the same pole with respect to them both and this pole is P.

THEOREM 2. *The singular points of the pencil of point conics determined by two given nondegenerate conics and the singular lines of the pencil of line conics determined by the same two conics are in one-to-one correspondence; a singular point and the corresponding singular line are pole and polar with respect to both conics.*

Let us next enumerate the singular points and singular lines of the various types of pencils.

Type	*Pencil of Point Conics* *Singular Points*	*Pencil of Line Conics* *Singular Lines*
I	Three	Three
II	Two	Two
III	One and a range *	One and a pencil
IV	One	One
V	A range	A pencil

We return now to the two pencils of Theorem 2. The theorem guarantees that there are just as many singular points associated with the one pencil as there are singular lines associated with the other. Interpreting the table in light of this fact, we conclude

* That is, a range of singular points and an isolated singular point.

THEOREM 3. *The pencil of point conics and the pencil of line conics which are determined by two given nondegenerate conics are of the same type.*

For example, if the pencil of point conics is of Type II, that is, has just two singular points, the pencil of line conics must have just two singular lines and hence must be of Type II.

Essentially equivalent to Theorem 3 is the following theorem.

THEOREM 4. *A pair of nondegenerate conics is of the same type whether the conics are considered as point conics or as line conics.*

This simple statement is rich in content. It says, for example, that, if two conics have four distinct points of intersection, they have four distinct common tangents, and conversely. Again, it tells us that, if the conics have two-point contact at a given point, they have two-line contact on their common tangent at the point, and vice versa. In fact, in the latter connection, it establishes for us the important proposition that k-point contact and k-line contact are equivalent.

THEOREM 5. *If two nondegenerate conics have k-point contact at a given point, they have k-line contact on their common tangent at the point, and vice versa.*

Following classical usage, we shall denote k-point contact or k-line contact as *contact of order* $k - 1$. Ordinary two-point or two-line contact is then contact of the first order.

Common Self-Conjugate Triangles. If two points have the same polars with respect to both the given conics, their line has the same pole with respect to them both. In other words:

CONTINUATION OF THEOREM 2. *The line joining two singular points is a singular line. The point of intersection of two singular lines is a singular point.*

If a triangle is to be self-conjugate with respect to both conics, its vertices and sides must be respectively singular points and singular lines of the two pencils determined by the conics. Inspection of our table shows immediately that a triangle of this description is possible when and only when the two pencils are of Type I or of Type III.

If they are of Type I, the one pencil has three singular points, and the other, three singular lines. The three singular points form a triangle, the diagonal triangle of the complete quadrangle determined by the four points of intersection of the two conics. The three singular lines

also form a triangle, the diagonal triangle of the complete quadrilateral determined by the four common tangents to the two conics. According to the Continuation of Theorem 2, these two triangles are one and the same triangle. Since the vertices and sides of this triangle are poles and polars with respect to both conics and no one of the vertices lies on either conic, each vertex and the opposite side are pole and polar and the triangle is self-conjugate with respect to both conics.

THEOREM 6. *If two nondegenerate conics intersect in four distinct points, or have four distinct common tangents, there exists a unique triangle which is self-conjugate with respect to both.*

In the case of two conics which have double contact (Fig. 27), there are ∞^1 common self-conjugate triangles. Each triangle has the point of intersection of the common tangents to the two conics as one vertex and two conjugate points on the chord of contact as the other two vertices.

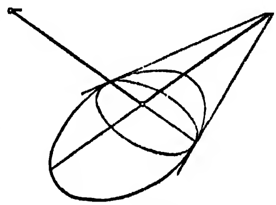


FIG. 27

Poles and Polars with Respect to a Pencil of Conics. The polars of a point r with respect to the conics of the pencil of point conics (5) are defined by (4). If r is a singular point of the pencil, the lines (2) are identical and equation (4) represents always the same line, no matter what the value of λ.* Otherwise, equation (4) represents a pencil of lines.

THEOREM 7 *a*. *The polars of a point with respect to the conics of a pencil of point conics form a pencil of lines, provided the point is not a singular point of the pencil of conics. A singular point has the same polar with respect to all the conics of the pencil.*

THEOREM 7 *b*. *The poles of a line with respect to the conics of a pencil of line conics form a range of points, unless the line is a singular line. A singular line has the same pole with respect to all the conics.*

It is evident from Theorems 6 and 7 that there is a unique triangle which is self-conjugate with respect to all the conics of a pencil of point conics of Type I, and that the same is true of a pencil of line conics of Type I. What are the corresponding facts for pencils of Type III?

* With one exception: the polar of the singular point r with respect to the degenerate conic to which r belongs is *undefined*.

The following interesting result is a direct consequence of Theorem 7 *b*.

THEOREM 8. *The centers of the central conics of a pencil of line conics lie, in general, on a line.*

EXERCISES

1. By means of the theory of poles and polars, give a direct proof that the diagonal triangle of the complete quadrangle determined by the four points of intersection of two nondegenerate conics is self-conjugate with respect to both conics. Show that the diagonal triangle is real except when two of the four points are conjugate-imaginary and the other two real; see Ex. 1, End of Chapter.

2. Draw carefully a figure for Theorem 6, constructing separately the two diagonal triangles mentioned in the proof.

3. The equations in point coordinates of a pair of nondegenerate conics of Type I or Type III can be reduced simultaneously to the forms

$$a_1x_1^2 + a_2x_2^2 + a_3x_3^2 = 0, \qquad b_1x_1^2 + b_2x_2^2 + b_3x_3^2 = 0.$$

Criticize this statement. When will the conics, as represented by the reduced equations, be of Type I?

4. Establish Theorem 2 directly in each of the cases II, III, IV, V, drawing a careful figure in each case. Prove that there is always at least one singular point on each singular line and at least one singular line through each singular point.

5. Prove Theorem 7 *b* and deduce from it Theorem 8. When do the exceptions to the latter theorem occur? Enumerate those which you find of special interest.

6. Deduce from Theorem 8 the fact that the mid-points of the diagonals of a complete quadrilateral are collinear.

7. Prove that the cross ratios of the polars of a nonsingular point with respect to four conics of a pencil of point conics are independent of the point, that is, depend only on the particular four conics chosen.

8. Show that the poles of a line with respect to the conics of a pencil of point conics constitute, in general, a nondegenerate conic. What are the exceptions?

Suggestion. Consider the pole of the line as the point of intersection of the polars of two points on the line and apply the theorem of the preceding exercise.

9. The centers of the central conics of a pencil of point conics lie, in general, on a nondegenerate conic. Prove this theorem and discuss the exceptions to it.

10. Show that the centers of the conics of the pencil determined by $y^2 = 2\,mx$ and $2\,xy = a$ lie on $y^2 + mx = 0$.

11. A pencil of Type III is self-dual. The centers of the central conics belonging to it lie, according to Theorem 8, on a line and, according to Ex. 9, on a conic. What are the facts? Discuss all possibilities.

12. State the duals of the theorems of Exs. 7 and 8.

16. Conjugate Points with respect to a Pencil of Point Conics. The points which are conjugate to a given point P with respect to all the conics of a pencil of point conics are the points common to all the polars of P with respect to the conics of the pencil. Hence, by § 15, Th. 7 *a*, we have

THEOREM 1. *Conjugate to a nonsingular point P with respect to all the conics of a pencil of point conics is a unique point P', the vertex of the pencil of polars of P. Conjugate to a singular point are all the points of the common polar of the singular point.*

If each of two conjugate points * is the only point conjugate to the other, neither can be a singular point. Moreover, neither can lie on the polar of a singular point; for, if one were on the polar of a singular point, the other would be this singular point.

COROLLARY. *The points which are neither singular points nor on the polars of singular points are conjugate in pairs. Conjugate to a non-singular point on the polar of a singular point is the singular point, and conjugate to a singular point is every point on the polar of the singular point.*

In the case of a pencil of Type I, for example, the singular points are the vertices, and their polars the opposite sides, of the common self-conjugate triangle. The points not on the triangle are conjugate in pairs. Conjugate to a vertex is every point on the opposite side, and conjugate to every point on a side, other than a vertex, is the opposite vertex.

Transformation of a Line. Let a point P trace a line L. What is the locus of the points P' which are conjugate to P?

The points P' are the points common to the polars p_1, p_2 of P with respect to two distinct conics C_1, C_2 of the pencil. As P traces L, p_1 and p_2 generate pencils of lines whose vertices are the poles, P_1 and P_2, of L with respect to C_1 and C_2. These pencils of lines are each projective with the range of points on L and hence are themselves projective. Consequently, the locus of the points P' common to pairs of corresponding lines p_1 and p_2 is, in general, a conic.

In discussing the locus in greater detail, we distinguish three cases.

A. P_1 and P_2 distinct and P_1P_2 not self-corresponding. The locus

* Here and throughout the paragraph we mean, by "conjugate points," points which are conjugate with respect to all the conics of the pencil.

is a nondegenerate conic. In this case L is not the polar of a singular point and does not contain a singular point. For, if L were the polar of a singular point, its poles P_1 and P_2 would coincide in the singular point. And if L, not itself the polar of a singular point, contained a singular point, the polars p_1 and p_2 of this singular point would be identical and P_1P_2 would be self-corresponding.

B. P_1 and P_2 distinct and P_1P_2 self-corresponding. The line L is not the polar of a singular point, but contains one singular point. The locus is a degenerate conic consisting of the line P_1P_2—the polar of the singular point on L—and a second line.

C. P_1 and P_2 coincident. The line L is the polar of a singular point, the point A in which P_1 and P_2 coincide. Both pencils have A as vertex, so that we now have a projective correspondence of a pencil of lines with itself. The fixed lines of this correspondence constitute the locus. On the other hand, these lines are the polars of the singular points on L. According as the correspondence is not or is the identity, just two or all the lines through A are fixed and just two or all the points on L are singular points. In the former case, the locus is the degenerate conic consisting of the polars of the two * singular points on L, whereas in the latter case it consists of all the points of the plane.

Inspection of these results justifies the following summary.

THEOREM 2. *If a point P traces a line L, not every point of which is singular, the points P' conjugate to P constitute a conic. The conic is nondegenerate unless L contains a singular point.*

The Eleven-Point Conic. On the conic which is the transform of a line L, there are several points of note. In enumerating them we restrict ourselves to the general case.

THEOREM 3. *If a pencil of point conics of Type* I *and a line L which contains neither a singular point nor a base point of the pencil* † *are given, the locus of the points conjugate to those of L is a nondegenerate conic C which passes through the following eleven points: (a) the three vertices of the common self-conjugate triangle; (b) the six points which are the harmonic conjugates of the points of intersection of L with the six sides of the basic quadrangle, with respect to the pairs of base points on these sides;*

* The two singular points on L may coincide. It can be shown that the doubly counting singular point thus obtained belongs to a multiply counting degenerate conic of the pencil.

† A base point of the pencil is a point common to all the conics of the pencil.

(c) the double points of the involution established on L by the pencil of conics.

The line L meets the sides of the self-conjugate triangle in three distinct points, and conjugate to these three points are the three vertices of the triangle. Hence C contains these vertices.

If B_1 and B_2 are two base points and P is the point in which their line meets L, the fourth harmonic point to B_1, B_2 and P is clearly conjugate to P with respect to all the conics of the pencil, and hence lies on C.

Since each of the double points of the involution on L is conjugate to the other, both double points lie on C.

EXERCISES

1. Give a direct proof of Theorem 2 for a pencil of Type I, using the description of the correspondence of conjugate points given in the text. Let L be in turn a side of the self-conjugate triangle, a line through a vertex other than a side, and a line not through a vertex. Then discuss the case in which L is a side of the basic quadrangle.

2. Show that a pencil of point conics of Type I which cuts the line at infinity in the involution whose double points are the circular points at infinity has the following properties: (*a*) the conics of the pencil, with the exception of two imaginary parabolas, are all *rectangular hyperbolas*; (*b*) the circle circumscribing the diagonal triangle of the complete quadrangle determined by the four points common to all the conics is the *nine-point circle* of each of the four triangles determined by these points; (*c*) this nine-point circle is a special case of the eleven-point conic.

3. Prove that the conic of Theorem 2, if it is nondegenerate, is identical with the conic of § 15, Ex. 8. Hence show that the locus of the centers of the rectangular hyperbolas of the pencil of Ex. 3 is the nine-point circle.

4. Prove that, if a triangle is self-conjugate with respect to a nondegenerate rectangular hyperbola, the circle circumscribing the triangle passes through the center of the hyperbola.

Suggestion. Find a degenerate rectangular hyperbola with respect to which the triangle is self-conjugate.

5. A complete quadrangle with finite vertices and diagonal points is given. Show that the three diagonal points and the six mid-points of the sides of the quadrangle lie on a conic.

6. State and prove Theorem 3 for an arbitrary pencil of point conics.

7. Discuss conjugate lines with respect to a pencil of line conics.

17. Simultaneous Invariants of Two Conics. If the collineation

$$x_i = \sum d_{ij}x'_j, \qquad (i = 1, 2, 3), \qquad \Delta = |d_{ij}| \neq 0,$$

carries the quadratic forms

(1) $$\sum a_{ij}x_ix_j, \qquad \sum b_{ij}x_ix_j, \qquad a_{ij} = a_{ji}, \qquad b_{ij} = o_{ji},$$

into the quadratic forms

$$\sum a'_{ij}x'_ix'_j, \qquad \sum b'_{ij}x'_ix'_j, \qquad a'_{ij} = a'_{ji}, \qquad b'_{ij} = a'_{ji},$$

then it carries the quadratic form

(2) $$\sum (k\, a_{ij} + l\, b_{ij})x_ix_j$$

into the quadratic form

$$\sum (k\, a'_{ij} + l\, b'_{ij})x'_ix'_j,$$

no matter what values are given to k and l.

By Ch. XIV, § 8, the discriminant of (2), namely

(3) $$|k\, a_{ij} + l\, b_{ij}| \equiv |a_{ij}|k^3 + S_1\, k^2l + S_2\, kl^2 + |b_{ij}|l^3,$$

is a relative invariant of weight two with respect to the group of collineations; that is, the equation

$$|k\, a'_{ij} + l\, b'_{ij}| \equiv \Delta^2 |k\, a_{ij} + l\, b_{ij}|$$

is an identity in k, l. Hence the coefficients of the terms in the expansion of the discriminant (3) as a cubic form in k, l are relative invariants of weight two.

The values of the coefficients S_1 and S_2 are readily found to be: *

(4) $$S_1 = \sum A_{ij}b_{ij}, \qquad S_2 = \sum a_{ij}B_{ij}.$$

Inasmuch as they bear on both the given forms (1), they are known as *simultaneous invariants* of these two forms.

We now assume that the two conics

$$\alpha \equiv \sum a_{ij}x_ix_j = 0, \qquad \beta \equiv \sum b_{ij}x_ix_j = 0$$

are nondegenerate and discuss the geometrical significance of the vanishing of the invariants S_1 and S_2.

Theorem 1. *A necessary and sufficient condition that there exist a triangle self-conjugate with respect to the conic $\alpha = 0$ and inscribed in the conic $\beta = 0$ is that $S_1 = 0$.*

Since S_1 is an invariant, it suffices to prove the theorem for a special choice of the coordinate system. Take as the triangle of reference a

* The coefficient of k^2l, for example, is

$$\begin{vmatrix} b_{11} & a_{12} & a_{13} \\ b_{21} & a_{22} & a_{23} \\ b_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & b_{12} & a_{13} \\ a_{21} & b_{22} & a_{23} \\ a_{31} & b_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} & b_{13} \\ a_{21} & a_{22} & b_{23} \\ a_{31} & a_{32} & b_{33} \end{vmatrix}$$

triangle $A_1A_2A_3$ which is self-conjugate with respect to $\alpha = 0$ and whose first two vertices A_1, A_2 lie on $\beta = 0$.* Then

$$\alpha \equiv a_{11}x_1^2 + a_{22}x_2^2 + a_{33}x_3^2, \qquad a_{11}a_{22}a_{33} \neq 0,$$
$$b_{11} = 0, \qquad b_{22} = 0,$$

and

$$S_1 = \sum A_{ij}b_{ij} = \sum A_{ii}b_{ii} = A_{33}b_{33}.$$

Since $A_{33} \neq 0$, $S_1 = 0$ if and only if $b_{33} = 0$, that is, when and only when A_3 lies on $\beta = 0$.

Since S_2 is the dual of S_1, the geometrical significance of $S_2 = 0$ is the dual of that of $S_1 = 0$.

THEOREM 2. *A necessary and sufficient condition that there exist a triangle self-conjugate with respect to the conic $\alpha = 0$ and circumscribed about the conic $\beta = 0$ is that $S_2 = 0$.*

According to Th. 1, $S_2 = 0$ is also the condition that there exist a triangle self-conjugate with respect to $\beta = 0$ and inscribed in $\alpha = 0$. Hence:

THEOREM 3. *There exists a triangle self-conjugate with respect to one conic and circumscribed about a second, if and only if there exists a triangle self-conjugate with respect to the second conic and inscribed in the first.*

Two conics which are related in this way are called *apolar.*

EXERCISES

1. Show that, if there is one triangle which is self-conjugate with respect to one of two nondegenerate conics and is inscribed in the other, there are infinitely many.

2. If $\alpha = 0$ is a degenerate point conic of rank 2 and $\beta = 0$ a nondegenerate conic, show that the singular point of $\alpha = 0$ lies on $\beta = 0$ if and only if $S_1 = 0$, and that the constituent lines of $\alpha = 0$ are conjugate with respect to $\beta = 0$ if and only if $S_2 = 0$. Prove further that, if the rank of $\alpha = 0$ is 1, S_1 always vanishes, and $S_2 = 0$ when and only when the line of $\alpha = 0$ is tangent to $\beta = 0$.

3. State conditions in terms of the invariants of two conics (*a*) that a point lie on a nondegenerate conic; (*b*) that two distinct points be conjugate with respect to the conic; (*c*) that the line joining two points be tangent to the conic.

EXERCISES ON CHAPTER XVI

1. The points common to the point conics of a pencil of Type I may be all real, two real and two conjugate-imaginary, or conjugate-imaginary in pairs. In the case of a pencil of Type II, the double point of intersection must be real, whereas the two simple points may be real or conjugate-imaginary. The two double points of intersection of a pencil of Type III may be real or conjugate-

* That there exists a triangle with these properties is readily established.

imaginary. The triple and simple point of intersection of a pencil of Type IV and the quadruple point of intersection of a pencil of Type V are necessarily real. Prove these propositions and exhibit an example under each case.

2. Find the equation of the ellipse which goes through the point $(-2, 0)$, is tangent to the axis of y at the origin, and has the line $x - 2iy + 1 = 0$ as an asymptote.

3. Find the nondegenerate parabola which has double contact with the ellipse $x^2 + 4y^2 = 100$ at the points $(8, 3)$ and $(-6, -4)$.

4. Find the parabola which has three-point contact with the parabola $y^2 = x$ at the point $(1, 1)$ and meets this parabola again at the origin.

5. In a quadratic equation in x and y the coefficient of the term in xy is arbitrary and all the other coefficients fixed. What does the equation represent?

6. Prove that, if each two opposite sides of a complete quadrangle are perpendicular, each vertex is the point of intersection of the altitudes of the triangle determined by the other three vertices. Show that the four points $(0, 0)$, $(1, 2)$, $(1, -2)$, $(-3, 0)$ determine a complete quadrangle of this type and that all the real conics through them are *rectangular hyperbolas.*

7. Prove that, if a pencil of point conics contains two rectangular hyperbolas, all its real conics are rectangular hyperbolas. Hence show that a pencil of Type I consists primarily of rectangular hyperbolas if and only if the four points common to all the conics determine a complete quadrangle of the type described in Ex. 6; it is assumed that the four points are finite.

8. A point conic is a rectangular hyperbola if and only if the point of intersection of the altitudes of an inscribed triangle lies on the conic.

9. A pencil of point conics which does not consist exclusively of circles contains at most one circle. A necessary and sufficient condition that it contain a circle is that it contain two conics which have their axes * parallel but are not similar and similarly placed. Prove these propositions.

10. Show that, if there are two conics in a pencil of point conics whose axes are parallel, the axes of each two conics in the pencil are parallel.

11. Using the results of Exs. 9, 10, devise a method of constructing the circle osculating a nondegenerate conic at a given point.

12. Show that, if from a point P on a nondegenerate conic tangents are drawn to a confocal conic, these tangents are equally inclined to the tangent at P.

13. Prove that the tangents from a point P to a central conic make equal angles with the lines joining P to the real foci.

14. A nondegenerate central conic is determined by three real tangents and a real focus. Show that the second real focus is the isogonal conjugate of the given focus with respect to the triangle formed by the three real tangents; see Ch. III, § 10, Ex. 5.

* To preserve simplicity of statement, we ascribe to a nondegenerate parabola, here and in Ex. 10, a second axis perpendicular to the actual axis.

15. A variable central conic always has a given point as a real focus and remains tangent to two real lines. Prove that the locus of the second real focus is a straight line.

16. One real focus of a variable central conic which is tangent always to three real lines traces a straight line. Show that the locus of the second real focus is, in general, a nondegenerate conic. What are the exceptions?

Suggestion. Use trilinear coordinates (Ch. X, § 7).

17. Prove that the locus of the pole of a given line with respect to a set of confocal parabolas is a line perpendicular to the given line, provided this line is a finite line other than a diameter common to all the parabolas.

18. Each two finite lines which are conjugate with respect to all the conics of a set of confocal central conics are mutually perpendicular. The lines which are conjugate to the lines of a pencil envelope a parabola which is tangent to the common axes of the conics, provided the vertex of the pencil is a finite point not lying on an axis. Prove these propositions.

19. Show that the pencils of tangents to the conics of a pencil of point conics at two simple intersections of all the conics are perspective. Hence prove that the locus of the centers of the conics of a pencil of similar and similarly placed, nonconcentric, central conics is a straight line.

20. A variable line conic moves so that it is always tangent to four given lines, no three concurrent. Show that the line joining its points of contact with two of the given lines always passes through a fixed point. Identify this point.

21. On a common tangent of a pencil of line conics two points, neither of which is an intersection of common tangents, are chosen. Show that the two pencils of tangents from these points to the conics of the pencil are projective.

22. Show that the locus of the focus of a variable parabola which is always tangent to three given nonconcurrent finite lines is a circle.

23. A variable point conic always passes through four given points, no three collinear. A triangle inscribed in the conic has one vertex at one of the given points and the two adjacent sides fixed. Discuss the envelope of the third side.

24. A pencil of point conics containing one circle is given and the isotropics through a simple finite intersection P of all the conics are drawn. Show that the envelope of the line joining the second points of intersection of these isotropics with the general conic of the pencil is a nondegenerate parabola with P as focus.

25. Show that the conics of a pencil of line conics of Type I are concentric if and only if the four common tangents form a parallelogram.

26. One asymptote of a variable central conic which circumscribes a given triangle always passes through a fixed finite point. Show that the second asymptote envelopes a conic which is inscribed in the triangle.

Suggestion. First prove analytically the projective generalization of the theorem.

CHAPTER XVII

APPLICATIONS OF TRANSFORMATIONS *

1. Polar Reciprocation with respect to an Arbitrary Conic. We have already noted, in Ch. X, § 6, that a correlation carries an arbitrary projective configuration into the dual configuration and is the most general transformation of the plane which has this property. Thus the correlations are the transformations which guarantee that, if a certain projective theorem is true, the dual theorem is also true.

It was Poncelet who first developed the principle of duality. He employed, not the general correlations, but the involutory correlations, the transformations of pole into polar with respect to nondegenerate conics (Ch. XIV, § 3). In giving an account of his method we employ his own terminology. Two figures which correspond by a transformation of poles and polars he called *reciprocal polar figures*, intending to convey by the word *reciprocal* the meaning we have attached to the word *dual*. The transformation itself is, then, known as *polar reciprocation*.†

One example suffices to show how polar reciprocation ensures the truth of a projective theorem when the dual theorem is known to be

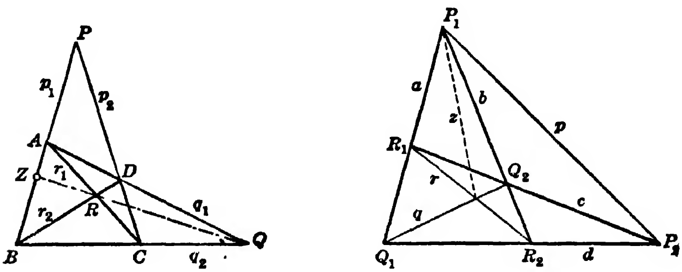


FIG. 1

true. Let the harmonic properties of the complete quadrilateral be given, and let it be required to establish those of the complete quadrangle. Reciprocate the given complete quadrangle with respect to a

* We shall find it convenient, in this chapter, to exclude conics without real traces.

† The name is particularly fitting: polar reciprocation is "polar dualization," that is, dualization by means of the theory of poles and polars.

fixed nondegenerate conic. The reciprocal polar figure is a complete quadrilateral. The vertices A, B, C, D of the quadrangle reciprocate into the sides a, b, c, d of the quadrilateral, the pairs of opposite sides p_1, p_2, q_1, q_2, r_1, r_2 of the quadrangle into the pairs of opposite vertices P_1, P_2, Q_1, Q_2, R_1, R_2 of the quadrilateral, and the diagonal points P, Q, R of the quadrangle into the diagonals p, q, r of the quadrilateral, as shown in Fig. 1.*

Not only is the quadrilateral the reciprocal of the quadrangle, but the quadrangle is the reciprocal of the quadrilateral. Consequently, from the harmonic properties of the quadrilateral follow those of the quadrangle. For example, the pole of the line z introduced in the quadrilateral is the point Z marked in the quadrangle. Since by hypothesis $(a\ b,\ p\ z) = -1$, it follows on reciprocation that $(A\ B,\ P\ Z) = -1$.

A Second Use of Polar Reciprocation. Suppose that the conic with respect to which we reciprocate the complete quadrangle is a central conic with P as center. The polar p of P is then the line at infinity. Hence a and b, and also c and d, are parallel, and the complete quadrilateral is a parallelogram. But the harmonic properties of a parallelogram are self-evident, and from them we obtain, on reciprocating back, the harmonic properties of the complete quadrangle.

This example illustrates a method by means of which polar reciprocation can frequently be used to establish a projective theorem. By proper choice of the conic of reciprocation the figure for the projective theorem is carried into a figure which pictures, not the dual theorem, but a special affine case of the dual theorem. If this special case of the dual theorem can be proved, the figure for it can be reciprocated back into the original figure and the required projective theorem is thus established.

A central conic is usually employed as the conic of reciprocation and the success of the method ordinarily depends on a wise choice of the center of this conic,—*the center of reciprocation.* See Ex. 3.

EXERCISES

1. Show that a complete quadrilateral can be reciprocated into a parallelogram and its diagonals. Hence establish the harmonic properties of the complete quadrilateral.

* The reader should draw his own figure and follow the process of reciprocation step by step. Practice in these simple cases is the only safeguard against confusion in the more complicated cases which are to follow.

2. Assuming one half of Desargues' triangle theorem, prove the other half by polar reciprocation.

3. Without assuming the dual theorem, prove that, if the corresponding sides of two triangles intersect in collinear points, the lines joining corresponding vertices are concurrent.

Suggestion. Take as the center of reciprocation the point of intersection of two of the lines joining corresponding vertices.

2. Continuation. Polar Reciprocation of Conics. In reciprocating a curve C, we can think of it either as a point locus C_P or as a line envelope C_L. The reciprocal of C_P is a line envelope C'_L, and the reciprocal of C_L is a point locus C'_P.*

The point curve C_P and the line curve C_L correspond in the sense of Ch. XIII, § 2; taken together, they form the curve C. Similarly, *the reciprocal line and point curves C'_L and C'_P correspond, and form together a curve C'.*

To establish this fact we must show that the tangents to C'_P are the lines of C'_L, and that the contact points of C'_L are the points of C'_P. Let T be an arbitrary point of C'_P, corresponding to the tangent t to C; T' a neighboring point of C'_P, corresponding to the tangent t' to C; and q the line joining T and T', corresponding to the point Q common to t and t'. The tangent p to C'_P at T is the limit of q as T' approaches T along C'_P. Since polar reciprocation is continuous, the pole of p must be the limit of Q as t' approaches t along C and is, therefore, the contact point P of C on t. Thus, the tangent p to C'_P is the polar of the point P of C. But p was an arbitrary tangent to C'_P. Consequently, the tangents to C'_P are the polars of the points of C and hence are the lines of C'_L. Similarly, the contact points of C'_L are the points of C'_P.

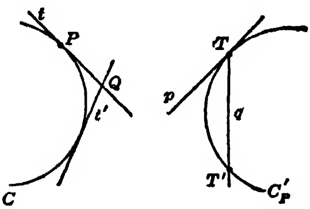


FIG. 2

As direct consequences of our argument, we have

THEOREM 1. *A point and corresponding tangent of a curve C reciprocate into a tangent and corresponding point of the polar reciprocal curve C'.*

THEOREM 2. *Tangent curves reciprocate into tangent curves.*

* If C has only one aspect, the situation is simple. For example, if C can be considered only as a point curve, its polar reciprocal can be thought of only as a line curve. Thus the reciprocal of a degenerate point conic is a degenerate line conic, and vice versa.

We turn now to polar reciprocation of conics. Inasmuch as a correlation is a linear transformation and preserves harmonic division, we conclude immediately:

THEOREM 3. *The polar reciprocal of a conic C is a conic C'. Two points conjugate with respect to C reciprocate into two lines conjugate with respect to C'. A point and a line which are pole and polar with respect to C reciprocate into a line and a point which are polar and pole with respect to C'.*

Since the polar reciprocal of a point conic is a line conic, a pair of point conics reciprocates into a pair of line conics, and a pencil of point conics into a pencil of line conics. The points common to the point conics correspond to the tangents common to the line conics, the degenerate conics of the pencil of point conics correspond to the degenerate conics of the pencil of line conics, and the singular points of the pencil of point conics correspond to the singular lines of the pencil of line conics.

The last of these facts leads, by inspection of the table of singular points and lines of Ch. XVI, § 15, to the following theorems.

THEOREM 4. *A pair or a pencil of point conics of a given type reciprocates into a pair or a pencil of line conics of the same type.*

THEOREM 5. *The order of contact of two conics is preserved by polar reciprocation.*

Example 1. Assuming Pascal's Theorem, prove Brianchon's Theorem by polar reciprocation.

Reciprocate the figure representing the hypothesis of Brianchon's Theorem. The result is a figure picturing the hypothesis of Pascal's Theorem. Construct in this figure the conclusion of Pascal's Theorem. Then the reciprocal construction in the original figure guarantees the conclusion of Brianchon's Theorem.

Example 2. Find the polar reciprocal of the parabola

$$y^2 = 2\,mx + m^2 \tag{1}$$

with respect to the circle

$$x^2 + y^2 = 1. \tag{2}$$

First Method. The equations of the polar reciprocation with respect to (2), expressed in homogeneous coordinates, are

$$\rho x_1 = u_1, \qquad \rho x_2 = u_2, \qquad \rho x_3 = -\,u_3.$$

Hence the parabola reciprocates into

$$u_2^2 + 2\,mu_1u_3 - m^2u_3^2 = 0.$$

This conic has, as its equation in point coordinates,

$$m(x^2 + y^2) + 2\,x = 0, \tag{3}$$

and so is a circle.

Second Method. The equation of the tangent to the parabola (1) at the point (x, y) is

$$-\,mX + yY - (mx + m^2) = 0.$$

Let the pole of this tangent with respect to the circle (2) be (x', y'). A second equation of the tangent is, then,

$$x'X + y'Y - 1 = 0.$$

Hence

$$-\frac{m}{x'} = \frac{y}{y'} = mx + m^2.$$

Substituting into (1) the values found for x and y from these equations, we find, as the polar reciprocal of the parabola, the circle (3).

EXERCISES

1. Do Example 1 in the text in detail, drawing and marking carefully the figures.

2. Assuming the following theorems, prove their duals by polar reciprocation: (*a*) Steiner's Theorem; (*b*) Ch. XVI, § 7, Th. 2; (*c*) Ch. XVI, § 7, Ex. 9; (*d*) Desargues' Involution Theorem.

3. Find the polar reciprocal of the hyperbola $xy = a^2$ with respect to the parabola $y^2 = 2\,mx$.

4. Find the polar reciprocal of the circle $x^2 + y^2 - 6\,x - 3 = 0$ with respect to the hyperbola $2\,x^2 - y^2 = 6$.

5. Show that a nondegenerate conic reciprocates with respect to a central conic into a hyperbola, parabola, or ellipse, according as the center of reciprocation is outside, on, or inside the conic.

6. What can you say of the polar reciprocal of a parabola with respect to a parabola?

7. Show that polar reciprocation with respect to the conic

$$x_1^2 + x_2^2 + x_3^2 = 0$$

has the equations

$$\rho x_1 = u_1, \qquad \rho x_2 = u_2, \qquad \rho x_3 = u_3.$$

Hence prove that, if in the equations constituting an analytic proof of a projective theorem point coordinates and line coordinates are everywhere interchanged, the new equations furnish a proof of the dual theorem.

3. Polar Reciprocation with respect to a Circle. Let the conic of reciprocation be a circle with center at O and radius r. The polar of a point P is, by symmetry, perpendicular to OP. Moreover, since OQP is a right triangle, $\overline{OP'} \cdot \overline{OP} = r^2$.

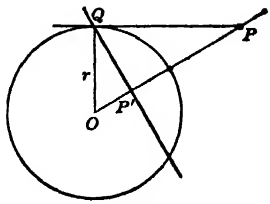


Fig. 3

Theorem 1. *The polar of a finite point P, other than O, is perpendicular to OP and at a distance OP' from O such that*

$$\overline{OP'} \cdot \overline{OP} = r^2.*$$

The theorem pictures clearly the relative positions of a point and its polar. It tells us, in particular, that the nearer the point is to O, the farther away from O is its polar, and vice versa.

Theorem 2. *The polars of the circular points at infinity, I and J, are the isotropics OI and OJ.*

For, the polars of I and J are the tangents at I and J, that is, the asymptotes OI and OJ of the circle.

Theorem 3. *A circle reciprocates into a nondegenerate conic which has O as a focus and the polar of the center of the circle as the corresponding directrix.*

The circle must in any case reciprocate into a conic. Since I and J are on the circle, the isotropics OI and OJ are tangent to the conic and O is consequently a focus of the conic.

The focus O and the corresponding directrix c are pole and polar with respect to the conic. Hence their reciprocals, o and C, are polar and pole with respect to the circle. But, since O is the center of reciprocation, o is the line at infinity. Thus the line at infinity and C are polar and pole with respect to the circle, and so C is the center of the circle.

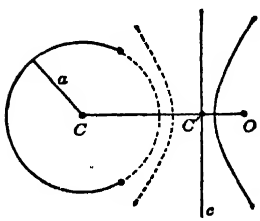


Fig. 4

A similar proof may be given for the converse:

Theorem 4. *The polar reciprocal of a nondegenerate conic with respect to a circle having its center at a focus is a circle whose center is the pole of the corresponding directrix.*

* The theorem has been proved when P is outside the circle. An analytic proof covering all cases can readily be given.

The applications of polar reciprocation with respect to a circle may be conveniently classified under three heads.

A. *Proof of Projective Theorems.* To establish a projective property of a conic it suffices to prove the dual property for a circle. Let it be required, for example, to show that four given tangents to an arbitrary nondegenerate conic cut an arbitrary fifth tangent in a constant cross ratio. Reciprocate the conic into a circle by the method of Th. 4. It suffices, then, to prove that a cross ratio of the four lines joining four points on the circle to an arbitrary fifth point is constant; see Ch. VI, § 4, Ex. 8.

B. *Reciprocation of Properties Involving Angle.* The applications under this head are based on the theorem pictured in Fig. 5, namely: *The angle which two points subtend at O is equal to one of the angles under which their polars intersect.*

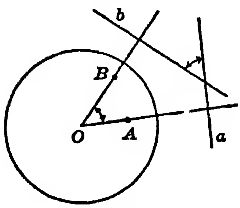


FIG. 5

Example 1. Generalize by polar reciprocation with respect to a circle: *A tangent to a circle is perpendicular to the radius drawn to the point of contact.*

By Th. 3, the circle reciprocates into a conic with O as focus and c as the corresponding directrix. The tangent t and the corresponding point P of the circle reciprocate into a point T and the corresponding tangent p of the conic, and the line s joining P to C goes into the point S in which p meets c. Since t and s are perpendicular, the points T and S subtend a right angle at O. Thus the required generalization is: *The segment of a tangent to a conic between the point of contact and a directrix subtends a right angle at the corresponding focus.*

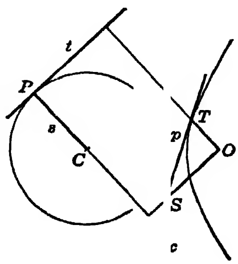


FIG. 6

Since Theorem 4 guarantees that every nondegenerate conic can be obtained from some circle by our special method of reciprocation, and since the given theorem is true for all circles, the generalized theorem is established for all nondegenerate conics.

Example 2. Generalize: The locus of the point of intersection of two tangents to a circle which cut under a constant angle is a concentric circle.

The figure tells the story. The generalized theorem is: The envelope of a chord of a nondegenerate conic which subtends a constant angle at a focus is a conic which has this focus and the corresponding directrix in common with the given conic.

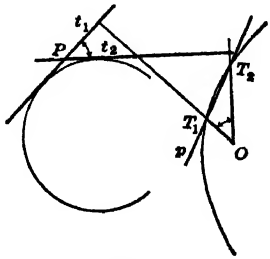


FIG. 7

C. *Reciprocation of Properties Involving Distances Measured from O.* Let it be required to generalize the fact that the sum of the distances from a point *within* a circle to two parallel tangents is constant, when the point is taken as the center O of the circle of reciprocation.

Since the tangents from O to the given circle are imaginary, the points at infinity on the reciprocal conic are imaginary and the conic is an ellipse (§ 2, Ex. 6). By hypothesis,

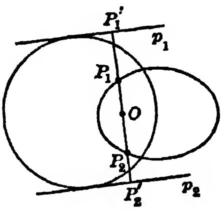


FIG. 8

$$OP_1' + OP_2' = 2a,$$

By Th. 1,

$$OP_1' \cdot OP_1 = r^2, \qquad OP_2' \cdot OP_2 = r^2.$$

Hence

$$\frac{1}{OP_1} + \frac{1}{OP_2} = \frac{2a}{r^2}.$$

The sum of the reciprocals of the segments of a focal chord of an ellipse is constant.

EXERCISES

Prove by polar reciprocation with respect to a circle:

1. Steiner's Theorem. 2. Theorem 1 *b* of Ch. XV, § 1.

Generalize, by polar reciprocation with respect to a circle, the following propositions.

3. An angle inscribed in a semicircle is a right angle.

4. The envelope of a chord of a circle of constant length is a concentric circle.

5. The polar of a point with respect to a circle is perpendicular to the line joining the point to the center of the circle.

6. Two tangents to a circle make equal angles with their chord of contact and equal angles with the line joining the center to their common point.

7. The envelope of the hypotenuse of a right triangle which is inscribed in

a circle and has the vertex of the right angle at a fixed point on the circle is the center of the circle. In reciprocating take the fixed point as O.

8. Discuss the problem taken up under C in the text when the point is taken (*a*) on the circle; (*b*) outside the circle.

9. If a is the radius of the given circle in Fig. 4 and d is the distance from its center C to the center of reciprocation O, show by means of Th. 1 that the distance m from the focus to the corresponding directrix of the conic is r^2/d, and that the semi-latus rectum of the conic is r^2/a. Hence show that the eccentricity e of the conic is d/a.

10. Using the data just deduced for the conic, namely

$$e = \frac{d}{a}, \qquad m = \frac{r^2}{d},$$

show that a *given* circle can be reciprocated into a nondegenerate conic of any shape and size.

11. Generalize by polar reciprocation with respect to a circle: Two common tangents to two equal circles are parallel to one another and to the line of centers. In reciprocating the fact that two circles are equal, employ the results of Ex. 9.

12. If two perpendicular tangents be drawn to a circle, the locus of their point of intersection is a concentric circle and the envelope of their chord of contact is also a concentric circle. Observe that the radii of these two circles can be readily computed. Obtain a new theorem by reciprocating with respect to a circle whose center is on the first circle.

4. Applications of Projections. Collineations are used, in much the same way as polar reciprocation, for two purposes: (*a*) to obtain from affine and metric theorems projective generalizations; (*b*) to establish projective theorems by reducing them to affine or metric theorems capable of simple proof.

In discussing these applications, we shall speak of projections rather than of collineations, and employ the terminology of projection rather than that of general transformation theory. Whether we interpret our statements in terms of the actual process of projection or as referring to collineations of the plane into itself or into a second plane, is immaterial.

In a projection, the fates of four lines, no three concurrent, are at our disposal. We can, then, surely dispose of one at pleasure. Hence:

THEOREM 1. *Any given line can be projected into the line at infinity.*

In order to prove by projection the harmonic properties of a complete quadrilateral, it suffices to show that a complete quadrilateral can always be projected into a parallelogram. How is this done?

Proofs of Projective Properties of Conics. Here we rely primarily on the following theorem.

THEOREM 2. *A nondegenerate conic and a point within it can be projected into a circle and its center.*

Since the given point M is within the conic, its pole m meets the conic in two conjugate-imaginary points. Choose a real projection which carries these two points into I and J. The conic projects into a conic through I and J, that is, into a circle. Moreover, since M and m are pole and polar with respect to the conic, their projections are pole and polar with respect to the circle. The projection of m is the line at infinity. Hence M goes into the center of the circle.

An equivalent to Theorem 2 is evidently: *A nondegenerate conic and a line intersecting it in conjugate-imaginary points can be projected into a circle and the line at infinity.*

Example 1. Let it be required to prove Pascal's Theorem by projection.

Project the given conic into a circle and the line joining the points of intersection of *two* pairs of opposite sides of the inscribed hexagon into the line at infinity. The projected hexagon (Fig. 9) has, then, two pairs of opposite sides, say 1, 4 and 2, 5, parallel, and it remains to prove that the sides 3, 6 are parallel.

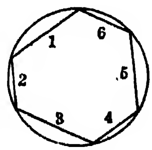


FIG. 9

Since sides 1 and 4 are parallel,

$$\text{arc } 2 + \text{arc } 3 = \text{arc } 5 + \text{arc } 6, \tag{1}$$

where we mean, for example, by arc 2, the shorter arc of the circle subtended by the chord 2. Similarly, since sides 2 and 5 are parallel,

$$\text{arc } 1 + \text{arc } 6 = \text{arc } 3 + \text{arc } 4. \tag{2}$$

Adding (1) and (2), we get

$$\text{arc } 1 + \text{arc } 2 = \text{arc } 4 + \text{arc } 5.$$

Hence the sides 3 and 6 are parallel.*

Example 2. If a variable triangle is circumscribed about a conic and two of its vertices trace fixed straight lines, find the locus of the third vertex.

We project the given conic and the point of intersection M of the

* The details of the proof are slightly different if any of the sides of the hexagon intersect within the circle.

given straight lines, which we assume does not lie on the given conic, into a circle and its center. Our problem is thus reduced to finding the locus of a vertex of a triangle which is circumscribed about a circle and whose other two vertices move on fixed lines passing through the center of the circle.

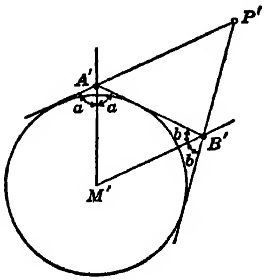


FIG. 10

This locus is a circle concentric with the given circle. For, the angle at P' (Fig. 10) is readily shown to be constant. In $\Delta\, A'B'P'$, $\angle\, A' = \pi - 2\,a$ and $\angle\, B' = \pi - 2\,b$, so that $\angle\, P' = 2(a + b) - \pi$. But $a + b$ in $\Delta\, A'B'M'$ is the supplement of the constant angle at M'.

A circle concentric with the given circle is a conic having double contact with it at I and J. Since I and J are the projections of the points in which the given conic is intersected by the polar of the point M, we obtain as the answer to our original problem:

The locus of the third vertex of the triangle is a conic which has double contact with the given conic at the points in which the polar of the point common to the two given lines meets the given conic.

Critique. Principle of Continuity of Analytic Proofs. According to Th. 2, we can project the given conic and the point M of Example 2 into a circle and its center only if M is a point *within* the given conic. Consequently, we have established the answer to our locus problem only in this case. It is reasonable to expect that the answer will be the same when M is *outside* the given conic. How shall we prove that this is true?

Again, in Example 1, we established Pascal's Theorem only in the case when the pole of the line joining the points of intersection of the two chosen pairs of opposite sides of the inscribed hexagon is a point M *inside* the given conic. How can the proof be extended so that it is valid also when M is outside the conic?

Poncelet has provided us with an effective means of meeting this difficulty in his *principle of continuity of analytic proofs.* This principle, stated in a form adapted to our needs, maintains that, if a projective theorem has been established in case a certain point M lies *inside* a certain conic, the analytic proof which it would then be possible to give for this case would be valid also when M is outside the conic.

The justification of the principle is contained in the following Lemma, the proof of which belongs to a course in Analysis.

LEMMA. *If a homogeneous algebraic function $f(r_1, r_2, r_3)$ of the coordinates (r_1, r_2, r_3) of a point M vanishes for all points M inside a certain conic, then it vanishes identically.*

Suppose that M is the point M of the proof of Pascal's Theorem and that we have deduced the equation whose truth, when $M : (r_1, r_2, r_3)$ is inside the given conic, establishes Pascal's Theorem analytically in this case. This equation is of the form $f(r_1\, r_2, r_3) = 0$, where $f(r_1, r_2, r_3)$ is a homogeneous algebraic function of r_1, r_2, r_3. Hence it follows, since $f(r_1, r_2, r_3) = 0$ is a true equation for all points M inside the given conic, that $f(r_1, r_2, r_3) \equiv 0$ and the theorem is always valid.

The answer to the locus problem of Example 2 involves two parts, first that the locus is a conic and, secondly, that this conic has double contact with the given conic in two specific points. The fact that the locus is a conic is expressed analytically by the equation $n - 2 = 0$,* which says that the degree n of the equation of the locus is 2. The contention that the locus has double contact with the given conic in two fixed points is given analytically by four equations which are of the forms $f_i(r_1, r_2, r_3) = 0$, $i = 1, 2, 3, 4$, where the functions $f_i(r_1, r_2, r_3)$ are homogeneous algebraic functions of r_1, r_2, r_3. Since the nature of the locus has been established when $M : (r_1, r_2, r_3)$ is inside the given conic, we know that all five equations are valid when M is thus restricted. Hence, by our Lemma, they are valid also when M is outside the given conic.

We must now qualify our general argument to cover certain contingencies which may arise. It is conceivable that the hypothesis of the proposition to be established may, by restrictions it places on M, limit the application of the Lemma or, indeed, exclude its use entirely. Suppose that the hypothesis restricts M by a condition which is equivalent to an inequality: $\phi(r_1, r_2, r_3) \neq 0$. Then the validity of the analytic proof is necessarily subject to the condition $\phi(r_1, r_2, r_3) \neq 0$, and the proposition cannot possibly be guaranteed by the Lemma for points M for which $\phi(r_1, r_2, r_3) = 0$. For example, the answer to our locus problem is not validated by the Lemma for points M *on* the given

* The expression $n - 2$ is a homogeneous polynomial in r_1, r_2, r_3 of degree 0.

conic. For, the very nature of the answer implies the tacit hypothesis that the polar of M meet the given conic in *distinct* points.*

EXERCISES

Prove by projection:

1. The triangle theorem of Desargues.
2. Steiner's Theorem.
3. The theorem of Ex. 1, End of Ch. XIV.
4. Brianchon's Theorem.
5. Theorem 2 of Ch. XVI, § 7.
6. The involution theorem of Desargues, for a pencil of Type I.
7. The pairs of points in which the lines of a pencil cut a conic form an involution.

Generalize by projection:

8. A chord of a circle which is tangent to a concentric circle is bisected by the point of contact.
9. If two circles are tangent at a finite point P and a line is drawn through P meeting the circles again in P_1 and P_2, the tangents to the circles at P_1 and P_2 are parallel.
10. A variable triangle is inscribed in a fixed conic and two of its sides turn about fixed points. Find by projection the envelope of the third side.

5. Continuation. Projection of Perpendicularity and Angle. We make use here of the projective interpretations of angle and perpendicularity due to Laguerre, Ch. VIII, § 6. From the interpretation of perpendicularity, we conclude that *two perpendicular lines project into two lines which separate harmonically the lines joining their common point to the projections of the circular points at infinity.*

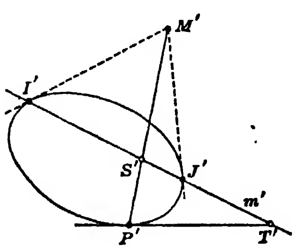


FIG. 11

Example 1. Let it be required to generalize by projection the theorem: The tangent to a circle is perpendicular to the radius drawn to the point of contact.

The circle projects into a conic. The center M of the circle and the line at infinity project into a point M' and a line m' which are pole and polar with respect to the conic, and I and J project into the points I' and J' in which m' meets the conic. A point P on

* If M lies on the given conic, the locus is a conic which has four-point contact with the given conic at M.

the circle projects into a point P' on the conic, the tangent PT into the tangent $P'T'$, and the radius PM into the line $P'M'$. Finally, since $(PM\ PT, PI\ PJ) = -1$, we have

$$(P'M'\ P'T', P'I'\ P'J') = -1 \quad \text{or} \quad (S'T', I'J') = -1.$$

If M is a point and m is its polar with respect to a nondegenerate conic, the tangent at an arbitrary point P of the conic and the line MP intersect m in conjugate points.

Since an arbitrary conic and a point within it can be projected into a circle and its center, the theorem is established is this case and hence, by the principle of continuity, in all cases.*

Example 2. Generalize by projection: The envelope of a chord of constant length of a circle is a concentric circle.

The circle, its center M and the line at infinity, and I and J project, as in the previous example, into a conic, a point M' and its polar m' with respect to the conic, and the points I' and J' in which m' meets the conic. A chord AB of the circle goes into a chord $A'B'$ of the conic. Since AB always subtends the same angle at the center M of the circle, it follows from Laguerre's interpretation of angle in terms of cross ratio, that $(MA\ MB, MI\ MJ)$ is constant. Consequently, $A'B'$ moves so that the cross ratio $(M'A'\ M'B', M'I'\ M'J')$ is constant. Since the envelope of AB is a circle having double contact with the first circle at I and J, the envelope of $A'B'$ is a conic which has double contact with the first conic at I' and J'.

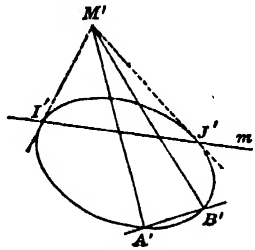


FIG. 12

The envelope of a chord of a conic which moves so that the lines which join its extremities to a fixed point M', not on the conic, form a constant cross ratio with the tangents from M', is a conic which has double contact with the given conic at the points in which it is met by the polar of M'.

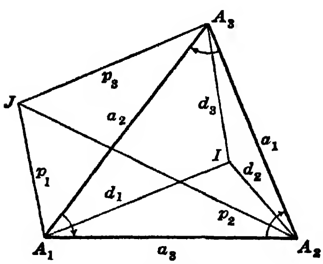


FIG. 13

Example 3. Generalize by projection: The sum of the three angles of a finite triangle is two right angles.

* The theorem is obvious but trivial if the point is on the conic.

Let θ_1, θ_2, θ_3 be the directed angles of the triangle, all measured in the same sense, as shown, for example, in Fig. 13. Then

$$(1) \qquad \theta_1 + \theta_2 + \theta_3 = \pm \pi.$$

If λ_1, λ_2, λ_3 are the cross ratios of the sets of four lines at the vertices of the triangle:

$$(2) \quad \lambda_1 = (a_2a_3,\, d_1p_1), \qquad \lambda_2 = (a_3a_1,\, d_2p_2), \qquad \lambda_3 = (a_1a_2,\, d_3p_3),$$

then, by Laguerre's interpretation of angle,

$$\theta_1 = \frac{1}{2\,i}\log\lambda_1, \qquad \theta_2 = \frac{1}{2\,i}\log\lambda_2, \qquad \theta_3 = \frac{1}{2\,i}\log\lambda_3,$$

and (1) becomes

$$\log\lambda_1 + \log\lambda_2 + \log\lambda_3 = \pm\, 2\,\pi\, i$$

or

$$\lambda_1\lambda_2\lambda_3 = e^{\pm 2\pi i}.$$

Since

$$e^{\phi i} = \cos\phi + i\sin\phi, \qquad e^{\pm 2\pi i} = \cos(\pm\, 2\,\pi) + i\sin(\pm\, 2\,\pi) = 1.$$

Hence

$$(3) \qquad \lambda_1\lambda_2\lambda_3 = 1.$$

Thus, equivalent to the fact that the sum of the angles is two right angles is the fact that the product of the three cross ratios (2) is unity. A remarkable relationship!

The desired generalization is now obvious: the equality (3) holds for any triangle and any pair of points neither of which lies on a side. We have thus established the direct theorem on which is based the theory of projective coordinates in the plane. The converse is readily proved from it.

EXERCISES

Generalize by projection:

1. An angle inscribed in a semicircle is a right angle.

2. The locus of a point from which the tangents to a circle form a constant angle is a concentric circle.

3. The polar of a point with respect to a circle is perpendicular to the line joining the point to the center.

4. The sum of the directed angles of a quadrilateral is equal to $\pm\, 2\,\pi$ radians.*

* How must "directed angle" be defined in order that this remains true when the quadrilateral is reentrant?

5. The locus of the point of intersection of two perpendicular tangents to a central conic is a circle whose center is the center of the conic.

EXERCISES ON CHAPTER XVII

Generalize by polar reciprocation with respect to a circle the following propositions.

1. The angles under which two intersecting circles meet are equal. If the angles are right angles, the tangents at the points of intersection pass through the centers of the circles.

2. The line joining the centers of two tangent circles goes through the point of contact and is perpendicular to the common tangent.

3. If a secant is drawn from a fixed point F outside a circle meeting the circle in the points A and B, the product $FA \cdot FB$ is constant. Take the fixed point as O.

4. If from a fixed point tangents are drawn to a set of concentric circles, their points of contact lie on a circle through the fixed point and the common center. Take the fixed point as O.

5. The sum of the distances from a focus to the points of contact of two parallel tangents to an ellipse is constant. Take the given focus as O.

6. The locus of the foot of the perpendicular dropped from a focus of a central conic on a variable tangent is the circle on the major axis as diameter. Take the given focus as O.

7. The locus of the point of intersection of perpendicular tangents to a central conic is a circle concentric with the conic. Take O (a) as a point on the circle; (b) as a focus of the conic.

8. The feet of the perpendiculars dropped from a point on a circle on the sides of a triangle inscribed in the circle are collinear. Take the point on the circle as O.

Ans. The circle which circumscribes the triangle formed by three tangents to a parabola goes through the focus.

Prove the following theorems by polar reciprocation with respect to a circle.

9. Two parabolas with the same focus and distinct axes have just one real finite common tangent and the angle which its points of contact subtend at the focus is equal to the angle between the axes.

10. A variable conic has a fixed focus and passes through two given points. Show that the locus of the point of intersection of the tangents at these points consists of two perpendicular lines through the focus. What can be said of the envelope of the directrix which corresponds to the given focus?

11. If two tangents to a parabola move always so that their points of contact subtend a right angle at the focus, the locus of their point of intersection is a rectangular hyperbola.

12. If the diagonals of a quadrilateral inscribed in a circle are perpendicular

and intersect in a fixed point, the four sides are tangent to a conic with the fixed point as a focus.

13. Two circles having imaginary finite points of intersection can be reciprocated into two confocal conics by taking, as the center of the circle of reciprocation, the center of one of the two null circles of the pencil of circles determined by the two given circles. Establish this fact, and hence prove that the contact points of a common tangent to the two circles subtend a right angle at the center of either of the null circles.

Generalize by projection the following theorems.

14. If two circles are tangent internally and the diameter of one is half that of the other, a chord of the larger drawn from the point of contact is bisected by the smaller.

15. The theorem of Ex. 2.

16. The locus of the point of intersection of perpendicular tangents to a parabola is the directrix.

17. The locus of the two points in which a chord of a circle is divided internally in a given ratio when the chord moves so that it always remains parallel to a diameter is an ellipse which has double contact with the circle at the extremities of the perpendicular diameter.

18. Two circles intersect in A and B. Through A any secant is drawn, cutting the two circles in C and D. The locus of the mid-point of CD is a circle through A and B with its center midway between the centers of the given circles.

19. The locus of the center of a variable circle which passes through a fixed point and is tangent to a fixed line is a parabola which has the fixed point as focus and the fixed line as directrix.

20. The theorem of Ex. 10.

Prove the following theorems by projection.

21. The theorem of Ch. II, § 6, Ex. 6.

22. An arbitrary tangent to one of three nondegenerate conics of a pencil of Type III, except a common tangent, is cut by the other two conics in four points whose cross ratio is constant.

23. If two triangles with distinct sides are circumscribed about a conic, their six vertices lie on a conic.

Suggestion. Reduce to the proposition which appears as the answer to Ex. 8.

24. If two triangles with distinct vertices are both self-conjugate with respect to a conic, their six vertices lie on a conic. See Ch. XVI, § 16, Ex. 4.

25. Show that each of two distinct equal circles has the same reciprocal polar curve with respect to the other if and only if the square of the distance between their centers is equal to three times the square of their common radius, and that the common reciprocal polar curve is then a hyperbola with the centers of the circles as foci and their common radius as major axis.

26. Prove that the three conics of the preceding exercise, the two circles and the hyperbola, have the property that each two are polar reciprocals with respect to the third.

27. Show that the polar reciprocal, with respect to $\Sigma a_{ij}x_ix_j = 0$, of $\Sigma b_{ij}x_ix_j = 0$ is $\Sigma c_{ij}x_ix_j = 0$, where

$$c_{ij} = \sum_{kl} B_{kl}a_{ik}a_{jl}.$$

28. Let A, B, C be three nondegenerate conics. Show that, if B and C are polar reciprocals with respect to A, and C and A are polar reciprocals with respect to B, then A and B are polar reciprocals with respect to C.

CHAPTER XVIII

THE CIRCLE

A. The Cartesian Geometry of the Circle

1. Introduction. We shall begin our study of the circle from the point of view of metric geometry considered as a subgeometry of projective geometry. We shall be interested only in real circles and their relationships to one another and to real points and real straight lines.

A real circle is the locus of an equation of the form

$$a_1(x_1^2 + x_2^2) + a_2x_1x_3 + a_3x_2x_3 + a_4x_3^2 = 0, \tag{1}$$

where x_1, x_2, x_3 are homogeneous Cartesian coordinates and a_1, a_2, a_3, a_4 are real constants, not all zero.

If $a_1 \neq 0$, (1) may be rewritten, in nonhomogeneous coordinates, as

$$\left(x + \frac{a_2}{2\,a_1}\right)^2 + \left(y + \frac{a_3}{2\,a_1}\right)^2 = \frac{(a, a)}{4\,a_1^2}, \tag{2}$$

where

$$(a, a) = a_2^2 + a_3^2 - 4\,a_1a_4. \tag{3}$$

We shall say that the circle (1) is *proper* if $a_1(a, a) \neq 0$, and *degenerate* if $a_1(a, a) = 0$.

If (1) is proper: $a_1 \neq 0$, $(a, a) \neq 0$, the square of the radius is

$$r^2 = \frac{(a, a)}{4\,a_1^2}.$$

Thus, a proper circle is either a *circle with a real trace*: $(a, a) > 0$, or a *circle without a real trace*: $(a, a) < 0$.

We distinguish three types of degenerate circles:

(*a*) *Null Circles*: $a_1 \neq 0$, $(a, a) = 0$. A null circle has a real finite center, but its radius is zero. It consists of the two isotropic lines through the center.

(*b*) *Finite Line Circles*: $a_1 = 0$, $(a, a) \neq 0$. Here the circle consists of a real finite line and the line at infinity.

(*c*) *Infinite Line Circle*: $a_1 = 0$, $(a, a) = 0$. The line at infinity, counted twice.

Though we shall devote our attention primarily to proper circles, and more particularly to those with real traces, we shall find the circles without real traces and the degenerate circles very useful, not only in enabling us to treat as special cases of a theorem what would otherwise be troublesome exceptional cases, but also in illuminating aspects of the theory which would otherwise be obscure.

It is readily proved analytically that two proper circles have in common, besides the circular points at infinity, two other points.* According as these two points are real and distinct, real and coincident, or conjugate-imaginary, we shall describe the circles as *intersecting*, *tangent*, or *nonintersecting*, inasmuch as they actually fit these descriptions in the domain of real points in which we are primarily interested.

Straight lines and points, other than those which, as component parts of a circle, are necessarily imaginary or infinite, shall be understood to be real and finite, unless the contrary is explicitly stated.

Exercise. Two proper circles, one or both of which are without real traces, are nonintersecting and, furthermore, can never be tangent at a finite point, real or imaginary. Why?

2. Power of a Point with respect to a Circle. Let C be a circle with a real trace with center O and radius r, and let P be a point. Through P draw a line meeting C in the points P_1, P_2. The product of the directed distances from P to P_1 and P_2 is independent of the line chosen, and is known as the *power*, p, of the point P with respect to C:

$$p = \overline{PP_1} \cdot \overline{PP_2}.$$

Evidently p is positive, zero, or negative, according as P lies outside, on, or inside C.

Taking the line P_1P_2 as the diameter through P, we find

$$p = OP^2 - r^2. \tag{1}$$

Theorem 1. *The power of a point with respect to a proper circle is equal to the square of the distance of the point from the center of the circle minus the square of the radius.*

The content of this statement is to be regarded as definition for a circle without a real trace. In this case, since $r^2 < 0$, p is always positive.

* These two points are finite, unless the circles are concentric. In this case the points coincide respectively with the circular points, each of which is then considered as counting twice.

Let C now be any proper circle and T the point of contact, real or imaginary, of a tangent drawn to it from P. If the tangent is not an isotropic line, the radius OT is perpendicular to it and, since the Pythagorean Theorem holds,

$$p = PT^2. \tag{2}$$

If the tangent is an isotropic, P is at O, T is at infinity and PT^2 does not exist. We agree to define PT^2 here as equal to $-r^2$, in order that (2) be universally valid.

Theorem 2. *The power of a point with respect to a proper circle is equal to the square of the tangent distance from the point to the circle.*

Exercise. Show that the power of a point inside a circle with a real trace, with respect to the circle, is equal to minus the square of half of the chord which is perpendicular at the point to the diameter through the point.

3. Orthogonal Circles. Points Inverse in a Circle. Let C_1, C_2 be two proper circles with centers O_1, O_2 and squares of radii r_1^2, r_2^2. If C_1, C_2 are not concentric, the relation

$$O_1O_2^2 = r_1^2 + r_2^2 \tag{1}$$

expresses the condition that radii drawn to a finite, real or imaginary, point of intersection be mutually perpendicular, and hence that C_1, C_2 be mutually orthogonal. If C_1, C_2 are concentric, we agree to take (1) as the definition of orthogonality.

In connection with (1), we note that $O_1O_2^2 - r_1^2$ is the power of O_2 with respect to C_1. Hence we obtain a theorem closely related to Theorem 2 of § 2.

Theorem 1. *A proper circle cuts a given proper circle orthogonally if and only if the square of its radius is equal to the power of its center with respect to the given circle.*

It follows that all the proper circles orthogonal to a circle without a real trace have real traces, whereas a proper circle orthogonal to a circle with a real trace may be of either type; see Ex. 3.

Points Inverse in a Circle. Two distinct points P_1, P_2 shall be said to be mutually inverse in a proper or finite line circle C if they are points of intersection of two proper circles C_1, C_2 which are orthogonal to C. The points P_1, P_2 are, by agreement (§ 1), real finite points. The circles C_1, C_2 have, then, real traces.

If C is a finite line circle, the centers of the two circles C_1, C_2 orthog-

onal to it lie on its finite line, and the points P_1, P_2 are symmetric in this line.

THEOREM 2. *Two distinct points are mutually inverse in a finite line circle C if and only if they are reflections of one another in the finite line of C.*

We proceed to show that, if C is a proper circle, the points P_1, P_2 are collinear with its center O. Let the line OP_1 intersect C_1 and C_2 again in P_2' and P_2'' respectively. Since C is orthogonal to C_1 and C_2, the square of its radius r^2 is equal to the powers of its center with respect to C_1 and C_2:

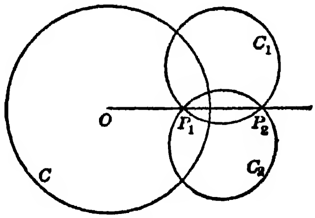


FIG. 1

$$r^2 = \overline{OP}_1 \cdot \overline{OP}_2' = \overline{OP}_1 \cdot \overline{OP}_2''.$$

Consequently P_2', P_2'' coincide, in P_2, and O, P_1, P_2 are collinear.

THEOREM 3. *Two distinct points are mutually inverse in a proper circle with center O and square of radius r^2 if and only if they are collinear with O and*

$$\overline{OP}_1 \cdot \overline{OP}_2 = r^2. \tag{2}$$

The necessity of the condition follows from the preceding argument. Conversely, equation (2), in connection with the fact that P_1, P_2 are collinear with O, says that *every* proper circle through P_1, P_2 is orthogonal to C.

THEOREM 4. *Two distinct points are mutually inverse in a proper circle C if and only if they are collinear with the center of C and one, and hence every, proper circle through them is orthogonal to C.*

This theorem follows directly from the preceding developments.

EXERCISES

1. Show that two points are mutually inverse in a proper circle if and only if they lie on a line through the center and separate harmonically the points, real or imaginary, in which this line meets the circle.

2. Prove directly that two circles without real traces can never be mutually orthogonal. Show that a circle without a real trace, with center O and square of radius r^2, is orthogonal to the circle with a real trace, with center O and square of radius $-r^2$.

3. There is a unique proper circle with a given point as center which is orthogonal to a given proper circle, provided the point does not lie on the given

circle. The circle always has a real trace, unless the given circle has a real trace and the given point lies within it.

4. A necessary and sufficient condition that a proper circle be orthogonal to a circle without a real trace, with center O and square of radius r^2, is that it intersect in diametrically opposite points the circle with a real trace with center O and square of radius $-r^2$, or coincide with this circle.

4. The Radical Axis of Two Circles. Coaxal Systems. If a point P moves so that it always has equal powers with respect to two proper circles, with centers O_1, O_2 and squares of radii r_1^2, r_2^2, we have, by § 2, (1),

$$\overline{O_1P}^2 - \overline{O_2P}^2 = r_1^2 - r_2^2.$$

If M is the foot of the perpendicular from P on O_1O_2, an equivalent relation is

$$\overline{O_1M}^2 - \overline{O_2M}^2 = r_1^2 - r_2^2.$$

But always

$$\overline{O_1M} - \overline{O_2M} = \overline{O_1O_2}.$$

Hence

$$\overline{O_1M} = \frac{\overline{O_1O_2}^2 + r_1^2 - r_2^2}{2\,\overline{O_1O_2}}$$

Evidently, $\overline{O_1M}$ is constant and M is a fixed point. Thus, the locus of P is the straight line through M perpendicular to O_1O_2.*

The argument breaks down when the given circles are concentric. But then the difference of the two powers of P has a constant value, not zero, and their ratio approaches unity when P recedes indefinitely, so that it is natural to take, as the locus, the line at infinity.

THEOREM 1. *The locus of a point which moves so that its powers with respect to two distinct proper circles are equal is a straight line.*

The line is called the *radical axis* of the two circles.

THEOREM 2 *a*. *The radical axis of two intersecting proper circles is their common chord, and that of two tangent proper circles, their common tangent.*

For, the radical axis is perpendicular to the line of centers and necessarily contains every finite point common to the two circles.

The radical axis of two nonintersecting circles intersects neither circle, that is, of course, in the real domain. Why?

* By retracing steps it can be proved that every point on the line actually belongs to the locus.

If two distinct points * P_1, P_2 are mutually inverse in each of two given proper circles, the circle on P_1P_2 as diameter cuts each of the given circles orthogonally (§ 3, Th. 4). By § 3, Th. 1, the center of this circle is the point M in which the radical axis of the given circles meets their line of centers, and the radius is the square root of the power of M. The power of M is, then, positive.

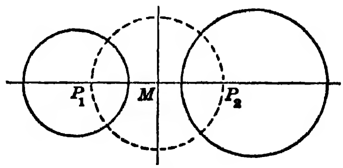


FIG. 2

According as two proper circles are intersecting, tangent, or concentric, the power of M is negative, zero, or undefined, and hence in these cases distinct points mutually inverse in the two circles fail to exist. On the other hand, if the two circles are nonintersecting with different centers, their radical axis intersects neither, the power of M is positive, and there are just two points P_1, P_2 mutually inverse in both circles.

THEOREM 3. *There exist two distinct points mutually inverse in each of two proper circles if and only if the circles are nonintersecting and have distinct centers.*

Incidentally, we have also obtained the complement of Theorem 2 *a*.

THEOREM 2 *b*. *The radical axis of two nonintersecting proper circles with distinct centers is the perpendicular bisector of the line-segment bounded by the two points which are mutually inverse in both circles, and the common power of the point M in which it meets the line of centers is equal to the square of the distance from M to either of these points. The radical axis of two concentric proper circles is the line at infinity.*

Coaxal Circles and Coaxal Systems. Of three proper circles C_1, C_2, C_3, let C_1, C_2 and C_1, C_3 have the same radical axis L. If L is a finite line, every point of it has the same power with respect to all three circles; if L is the line at infinity, the three circles are concentric. In both cases, L is also the radical axis of C_2, C_3.

THEOREM 4. *If, of the pairs of circles which may be formed from three proper circles, two pairs have the same line as radical axis, this line is also the radical axis of the third pair.*

The three circles are termed *coaxal*, and each is said to be coaxal with the other two.

* The reader will recall that by "points" we always mean real finite points unless the contrary is explicitly stated.

THEOREM 5. *A necessary and sufficient condition that a proper circle be coaxal with two proper circles is that it go through their points of intersection if they intersect, be tangent to them at their common point if they are tangent, share with them their common pair of inverse points if they are nonintersecting and have different centers, and be concentric with them if they are concentric.*

We give the proof in the only difficult case. Let C_1, C_2 be two nonintersecting proper circles with distinct centers, and let P_1, P_2 be the two points mutually inverse in both. Suppose that C is an arbitrary proper circle in which P_1, P_2 are inverse. Then, since P_1, P_2 are mutually inverse in both C and C_1, C and C_1 are nonintersecting circles with different centers, by Th. 3. Consequently, Theorem 2 *b* can be applied and guarantees the same radical axis for C and C_1 as for C_1 and C_2. Conversely, if C is coaxal with C_1 and C_2, C and C_1 are nonintersecting circles with distinct centers, since otherwise C_1 and C_2 would not be. Moreover, by Th. 2 *b*, the line of centers of C and C_1 is the line of centers of C_1 and C_2, and the point M in which the common radical axis meets this line has the same positive power with respect to all three circles. But, again by Th. 2 *b*, the two points mutually inverse in C and C_1 lie on the line of centers at a common distance from M equal to the square root of the power of M, and hence must be the points P_1, P_2 mutually inverse in C_1 and C_2.

The totality of all proper circles coaxal with two proper circles is called a *coaxal system*. From Theorem 5 we have:

THEOREM 6. *There are coaxal systems of four descriptions: (A) the circles through two points; (B) the circles tangent to a straight line at a given point; (C) the circles in which two points are mutually inverse; and (c) the circles with a given point as center.*

It is obvious that all the circles of a system of type (*A*) or (*B*) have real traces. It is equally evident that a system of type (*c*) contains circles of both kinds. This is true also of a system of type (*C*):

THEOREM 7. *Each point collinear with two given points, other than the points themselves, is the center of a unique proper circle in which the given points are mutually inverse. The circle has, or has not, a real trace according as its center is exterior or interior to the line-segment bounded by the given points.*

The proof of the theorem we leave to the reader.

EXERCISES

1. Give the proof of Theorem 5 in the first case.

2. Establish Theorem 7. Plot accurately a large number of the circles with real traces which belong to a coaxal system of type (C).

3. Prove that the mid-points of the four common tangents to two mutually external circles with real traces are collinear.

5. Continuation. Analytic Treatment. The power of the point $P:(x, y)$ with respect to the proper circle

$$(x - x_0)^2 + (y - y_0)^2 = r^2$$

is, by § 2, (1),

$$p = (x - x_0)^2 + (y - y_0)^2 - r^2.$$

THEOREM 1. *The power of the point (x, y) with respect to the proper circle*

$$\alpha(x, y) \equiv x^2 + y^2 + a_2x + a_3y + a_4 = 0$$

is $\alpha(x, y)$.

The content of the theorem is to be taken as the definition of the power of an imaginary point.

A complex point (x, y) has, then, the same power with respect to

$$\beta(x, y) \equiv x^2 + y^2 + b_2x + b_3y + b_4 = 0$$

as it has with respect to $\alpha = 0$ when and only when $\alpha(x, y) = \beta(x, y)$.

THEOREM 2. *The equation of the radical axis of the proper circles $\alpha = 0$ and $\beta = 0$ is $\alpha - \beta = 0$.**

Coaxal Systems. Assume that, in conjunction with $\alpha = 0, \beta = 0$, a third proper circle,

$$\gamma(x, y) \equiv x^2 + y^2 + c_2x + c_3y + c_4 = 0,$$

is given. The radical axis, $\alpha - \gamma = 0$, of $\alpha = 0$ and $\gamma = 0$ is the same as the radical axis, $\alpha - \beta = 0$, of $\alpha = 0$ and $\beta = 0$ if and only if two constants m, n, neither zero, exist so that

$$m(\alpha - \beta) + n(\alpha - \gamma) \equiv 0,$$

* When the two circles are concentric, $a_2 = b_2$, $a_3 = b_3$, and the equation $\alpha - \beta = 0$ has a locus only after homogeneous coordinates have been introduced in it.

In this connection, imagine $\alpha = 0, \beta = 0$ themselves written in homogeneous coordinates. Then the equation

$$\alpha - \beta \equiv [(a_2 - b_2)x_1 + (a_3 - b_3)x_2 + (a_4 - b_4)x_3]\,x_3 = 0$$

has as its locus the radical axis and the line at infinity. Hence, from the point of view of the projective geometry of conics (Ch. XVI, § 7), *the radical axis is the common chord of the two circles which is opposite to the line at infinity.*

or

$$n\gamma \equiv (m+n)\alpha - m\beta.$$

THEOREM 3. *A necessary and sufficient condition that a proper circle be coaxal with two given proper circles is that it be linearly dependent on them.*

Thus the system of circles coaxal with $\alpha = 0$, $\beta = 0$, as defined in § 4, consists of the *proper* circles in the totality of circles,

$$k\alpha + l\beta = 0,$$

linearly dependent on $\alpha = 0, \beta = 0$. We now extend the definition so that the coaxal system will include *all* the circles of this totality. In other words, we agree henceforth to think of a coaxal system as identical with the totality of circles, proper and degenerate, which are linearly dependent on two proper circles.

One degenerate circle of the coaxal system consists of the radical axis $\alpha - \beta = 0$ and the line at infinity.* This is evidently the only line circle in the system.

If $a_2 = b_2$ and $a_3 = b_3$, the circles of the system, other than the line circle, are precisely the circles with the point $(a_2, a_3, -2)$ as center. Hence a system of type (*c*) contains one null circle.

If $a_2 \neq b_2$ or $a_3 \neq b_3$, the system is of type (A), (B), or (C). The centers of its circles, other than the line circle, are readily seen to be precisely the finite points of the range of points $(k\,a_2 + l\,b_2, k\,a_3 + l\,b_3, -2(k+l))$. Hence there are as many null circles in the system as there are finite points of this range which do not serve as centers of proper circles of the system. Inspection of the various cases shows that there are no, one, or two null circles according as the system is of type (A), (B), or (C). The centers of the null circles of a system (C) are the two points mutually inverse in the proper circles of the system, and the center of the null circle of a system (B) is the point of contact of the circles of the system.

THEOREM 4. *Every coaxal system contains just one line circle, consisting of the radical axis and the line at infinity. The centers of the remaining circles are the finite points of a line perpendicular to the radical axis, except in the case of a system* (c), *when they all coincide.*

The four types of coaxal systems are shown in Fig. 3.

* See the preceding footnote.

THEOREM 5. *The locus of a point, real or imaginary, which moves so that the ratio of its powers with respect to two proper circles is constant is a circle * coaxal with the two circles, and conversely.*

This important criterion for coaxal circles is a consequence of Theorems 1 and 3.

Pencils of Circles. If

$$\alpha \equiv a_1(x_1^2 + x_2^2) + a_2x_1x_3 + a_3x_2x_3 + a_4x_3^2 = 0,$$
$$\beta \equiv b_1(x_1^2 + x_2^2) + b_2x_1x_3 + b_3x_2x_3 + b_4x_3^2 = 0,$$

are two distinct circles, the totality of circles

$$k\,\alpha + l\,\beta = 0$$

is known as a pencil of circles.

If $a_1 = 0$ and $b_1 = 0$, all the circles of the pencil are line circles, and conversely. The pencil consists of a pencil of lines with the line at infinity adjoined to each line to form a circle. Hence, it is of one of the types (a), (b) schematically pictured in Fig. 3.

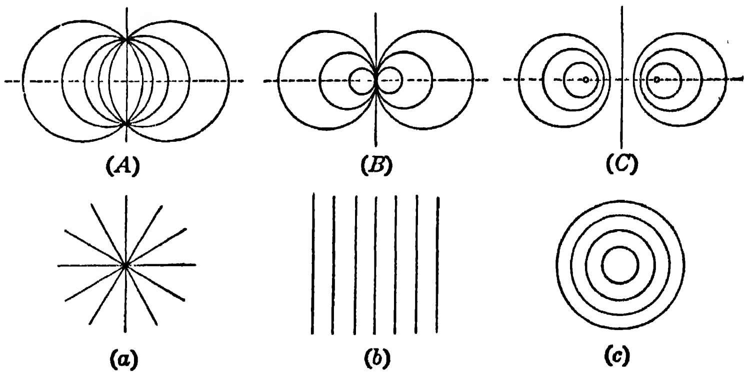


FIG. 3

THEOREM 6. *A pencil of circles is either a coaxal system or a pencil of line circles.*

In proving this theorem we shall make use of the following lemma (see Ch. XVI, § 4, Th. 1).

LEMMA. *Any two distinct circles of a pencil can be taken as the base circles* $\alpha = 0, \beta = 0$.

We have seen that if $a_1 = 0$ and $b_1 = 0$, the pencil is a pencil of line

* Strictly speaking, the locus consists of the finite points of the circle.

circles. If a_1, b_1 are not both zero, it can be shown that the pencil contains only a finite number of degenerate circles and hence an infinite number of proper circles. If two of the proper circles are taken as the base circles, the pencil becomes the totality of circles linearly dependent on two proper circles and so is a coaxal system.

EXERCISES

1. Show that a necessary and sufficient condition that the circle described on the line-segment bounded by two points collinear with the centers of two given proper circles as a diameter be coaxal with these circles is that the ratio of the powers of one point with respect to them be equal to the ratio of the powers of the other.

2. Every circle in which two given points are mutually inverse is the locus of a point, real or imaginary, which moves so that the ratio of the squares of its distances to the given points is constant, $\neq 0$, and conversely.

3. Through a point common to all the circles of a coaxal system (A) two lines, other than the radical axis, are drawn. Show that the ranges of their second points of intersection with the circles are projective, in that corresponding distances have a fixed ratio.

Suggestion. Use similar triangles.

4. A straight line meets each of two circles with real traces in distinct points. Show that the tangents to the one circle at its points of intersection with the line meet the tangents to the other circle at its points of intersection with the line in four points which lie on a circle coaxal with the given circles.

Suggestion. Employ Theorem 5, first applying the law of sines to an arbitrary triangle formed by the given line and two tangents, one to each circle.

5. If the vertices of a complete quadrangle lie on a circle with a real trace, a line which forms with one pair of opposite sides an isosceles triangle forms with each pair of opposite sides an isosceles triangle, provided it does not contain a diagonal point. A circle can then be drawn tangent to any chosen pair of opposite sides at their points of intersection with the line. The three circles of this description and the given circle are coaxal.

6. The *radical circle* of two proper circles is the locus of a point, real or imaginary, which moves so that its powers with respect to the given circles are negatives of one another. Show that the radical circle is coaxal with the given circles and has its center in the point halfway between their centers. When is it a circle with a real trace? A null circle? A circle without a real trace?

7. Show that, if the radical circle has a real trace, it is the locus of the centers of circles which cut one of the two given circles orthogonally and are cut by the other in diametrically opposite points.

8. A coaxal system (A) consists of all the circles through two real finite points; show that a coaxal system (C) consists of all the circles through two conjugate-imaginary finite points. The system (C) contains two real null circles; show that the system (A) contains two conjugate-imaginary null

circles. Prove that both systems are of Type I in the sense of Ch. XVI, § 4 and that both contain three distinct, real or imaginary, degenerate circles, two of which are null circles and the third a line circle.

Show that a coaxal system (*B*) is of Type II and that, of its three degenerate circles, the two null circles coincide. Of what type is a coaxal system (*c*)?

6. Orthogonal Circles and Orthogonal Systems. We consider first the proper circles orthogonal to two proper circles. From § 3, Th. 1, we conclude at once:

THEOREM 1 *a*. *A proper circle is orthogonal to two nonconcentric proper circles if and only if its center is a point on their radical axis and the square of its radius is the common power of the point with respect to them.*

We add a second criterion of a different nature.

THEOREM 1 *b*. *A necessary and sufficient condition that a proper circle be orthogonal to two nonconcentric proper circles is that it go through the two points mutually inverse in both if they are nonintersecting, cut them at right angles at their common point if they are tangent, and have their common points as inverse points if they intersect.*

The second part of the theorem follows from Theorem 1 *a*. The other two parts are direct consequences of the theory of orthogonal circles and inverse points. For example, according to § 3, Th. 4, a circle orthogonal to two nonintersecting proper circles with distinct centers must cut the line of centers in two points mutually inverse in both, and conversely.

The two theorems tell us that the proper circles orthogonal to our two circles are the proper circles of a coaxal system. Hence we are led to announce

THEOREM 2. *All the circles orthogonal to two given proper nonconcentric circles constitute a coaxal system; they are orthogonal also to all the circles coaxal with the given circles.*

THEOREM 3. *Orthogonal to all the circles of a coaxal system of nonconcentric circles are the circles of a second coaxal system. The two systems either are of types* (*A*) *and* (*C*) *or both are of type* (*B*). *The line of centers of the one is always the radical axis of the other, and the centers of the null circles of the one, when they exist, are the points common to all the circles of the other.*

That these theorems are true for all the circles involved, degenerate as well as proper, will be evident once we have extended our criterion

for the orthogonality of proper circles so that it is applicable to all circles.

The criterion for proper circles, to the effect that the square of the distance between the centers be equal to the sum of the squares of the radii, when expressed in terms of the coefficients in the equations,

$$\alpha \equiv a_1(x_1^2 + x_2^2) + a_2x_1x_3 + a_3x_2x_3 + a_4x_3^2 = 0,$$
$$\beta \equiv b_1(x_1^2 + x_2^2) + b_2x_1x_3 + b_3x_2x_3 + b_4x_3^2 = 0,$$

becomes

$$a_2b_2 + a_3b_3 - 2\,a_1b_4 - 2\,a_4b_1 = 0.$$

We agree to extend the validity of the criterion in this form to all circles.

Definition. *The two circles, $a : \alpha = 0$ and $b : \beta = 0$, are mutually orthogonal if and only if*

$$(a, b) = 0,$$

where

$$(a, b) \equiv a_2b_2 + a_3b_3 - 2\,a_1b_4 - 2\,a_4b_1.$$

The geometrical significance of the definition in the various possible cases is readily established.

Theorem 4. *Only when the center of a null circle lies on a given circle is the null circle orthogonal to the given circle. A finite line circle is orthogonal to a proper or null circle only if it contains the center of this circle. Two finite line circles are orthogonal only when their finite lines are perpendicular. The infinite line circle is orthogonal to every line circle and to no other circle.*

Inspection now shows that Theorems 2, 3 are true in every detail. In fact, we may go further and generalize them.

Theorem 5. *Orthogonal to two distinct circles, and hence to all the circles of their pencil, are all the circles of a second pencil.*

For a circle s which is orthogonal to two given circles a and b, we have

$$-\,2\,a_4s_1 + a_2s_2 + a_3s_3 - 2\,a_1s_4 = 0,$$
$$-\,2\,b_4s_1 + b_2s_2 + b_3s_3 - 2\,b_1s_4 = 0.$$

Since a and b are distinct, the rank of this system of two linear homogeneous equations in the four unknowns s_1, s_2, s_3, s_4 is two. Consequently, the solutions of the system are linearly dependent on two nonproportional solutions. Geometrically: the circles orthogonal to a and b are

linearly dependent on two distinct circles, that is, form a pencil. On the other hand, if $(a, s) = 0$ and $(b, s) = 0$, then $k(a, s) + l(b, s) = 0$ or $(k\,a + l\,b, s) = 0$, no matter what values k, l have. Hence each circle s orthogonal to a and b is orthogonal to every circle $k\,a + l\,b$ of their pencil.

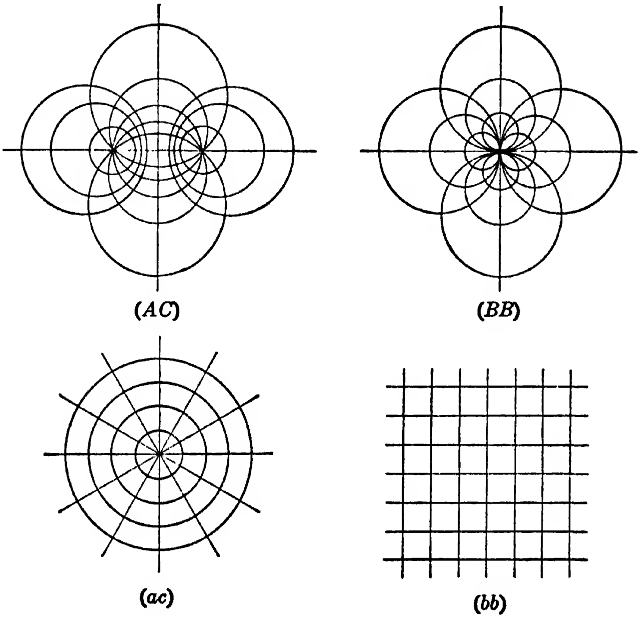


FIG. 4

Two pencils related as in Theorem 3 or 5 are said to be *conjugate* to one another and to form together an *orthogonal system*. The four different types of orthogonal systems are shown in Fig. 4.

Circles Orthogonal to Three Circles. The basis of the geometrical discussion in this case is the following theorem.

THEOREM 6. *The radical axes of the pairs of circles formed from three noncoaxal proper circles go through a point, which is finite or at infinity according as the centers of the three circles are not, or are, collinear.*

Since the given circles are not coaxal, the three radical axes are distinct (§ 4, Th. 4). If the centers of the three circles are collinear, the

radical axes are parallel, or two are parallel and the third is the line at infinity. Conversely, if the radical axes concur in a point at infinity, the centers lie on a line perpendicular to the finite radical axes. Hence, if the centers are not collinear, the radical axes cannot concur in an infinite point. There is, therefore, a finite point which is common to two of the radical axes; but this point evidently has the same power with respect to all three circles and so lies on the third radical axis.

The point common to the radical axes is known as the *radical center* of the three circles.

COROLLARY. *If the radical center is finite, it has the same power with respect to all three circles and lies inside, on, or outside all three.**

The reader can now readily establish

THEOREM 7. *There is a unique circle orthogonal to three noncoaxal proper circles. It is a finite line circle if the radical center is at infinity. If the radical center is finite, it is a circle with a real trace, a null circle, or a circle without a real trace according as the radical center is outside, on, or inside the given circles.*

We prove the generalization:

THEOREM 8. *There is a unique circle orthogonal to three circles which do not belong to the same pencil.*

Let a, b, c be the given circles. Since they are linearly independent, by hypothesis, the rank of the matrix of the coefficients in their equations is three. Hence the rank of the matrix of the three linear homogeneous equations,

$$(a, s) = 0, \qquad (b, s) = 0, \qquad (c, s) = 0,$$

in the four unknowns s_1, s_2, s_3, s_4, is three. Therefore, s_1, s_2, s_3, s_4, not all zero, are determined to within a factor of proportionality, and the theorem is proved.

THEOREM 9. *If there exists more than one circle orthogonal to three circles, the three circles belong to a pencil.*

EXERCISES

Prove:

1. Theorem 1 *b*, third part. **2.** Theorem 4. **3.** Theorem 7.

4. Give a construction, based on Theorem 6, for the radical axis of two nonintersecting circles with real traces and distinct centers.

* A point is outside, on, or inside a proper circle according as its power with respect to the circle is positive, zero, or negative. This theorem for circles with a real trace we take as definition for circles without a real trace; every point is, then, outside a circle of this type.

5. The three coaxal systems determined by the pairs of circles formed from three noncoaxal proper circles are all of type (C). Show that the centers of the six null circles in them lie on a circle.

6. Show that there is a unique circle in a pencil which is orthogonal to a given circle not belonging to the conjugate pencil.

7. Prove that, if there is a unique circle in a pencil orthogonal to all the circles of a second pencil, there is a unique circle in the second pencil orthogonal to all the circles of the first.

8. Two circles are linearly dependent only if they coincide; three, only when they belong to the same pencil; and four, only when they admit a common orthogonal circle. Five circles are always linearly dependent. Prove these propositions.

9. Prove that four distinct circles are linearly dependent if and only if, when they are paired, the pencils determined by the two pairs have a circle in common.

10. The coaxal systems determined by two distinct pairs of proper circles c_1, c_2 and c_3, c_4 have a circle in common. Show that if c_1, c_3, and also c_2, c_4, are intersecting circles, the four points of intersection are concyclic.

7. Centers of Similitude of Two Circles. A homothetic transformation,

$$(1) \qquad x' = \rho x + a, \qquad y' = \rho y + b, \qquad \rho \neq 0,$$

other than the identity, is *i*) a radial transformation of similarity ($\rho > 0, \neq 1$), or *ii*) a radial transformation of similarity (or the identity) followed by a reflection in a point ($\rho < 0$), or *iii*) a translation ($\rho = 1$). In cases *i*) and *ii*), the finite fixed point, or center, has the homogeneous coordinates $(a, b, 1 - \rho)$.*

A transformation *ii*) may equally well be described as a radial transformation of similarity with *negative* ratio of similitude, and a translation *iii*) as a degenerate radial transformation with center at the point at infinity $(a, b, 0)$ in the direction of translation and ratio of similitude unity. Accordingly, we agree to call the point $(a, b, 1 - \rho)$ the *center*, and ρ the *ratio of similitude*, of an arbitrary transformation (1), other than the identity.

If there is to be a (real) homothetic transformation which carries a given proper circle C into a prescribed proper circle C', both circles

* All these facts are evident once we note that, if $\rho \neq 1$, (1) can be rewritten in the form

$$x' - x_0 = \rho(x - x_0), \qquad y' - y_0 = \rho(y - y_0),$$

where (x_0, y_0) is the point with homogeneous coordinates $(a, b, 1 - \rho)$.

must have real traces or both must be without real traces. We restrict ourselves to the former case as the one of greater interest.

If C and C' are mutually external, it is geometrically evident (Fig. 5) that there are two homothetic transformations which carry C into C'. Their centers are respectively the points of intersection of the direct and transverse common tangents and their ratios of similitude are r'/r and $-r'/r$, where r and r' are the radii of C and C'.

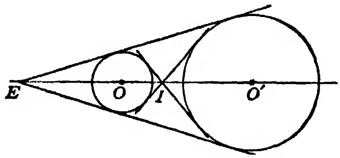


FIG. 5

In any case, a homothetic transformation which carries C into C' necessarily carries the center O of C into the center O' of C', and has as its ratio of similitude r'/r or $-r'/r$. Its center must then divide the line-segment OO' in the ratio r/r' or $-r/r'$, if O and O' are distinct, and coincide with O and O' if they are identical.

THEOREM 1. *There are always just two homothetic transformations which carry a given circle with a real trace into a second given circle with a real trace.*

The center of the transformation with positive ratio of similitude is called the *external center of similitude*, and that of the transformation with negative ratio, *the internal center of similitude*, of the two circles.

THEOREM 2. *The external and internal centers of similitude of two nonconcentric circles C, C' divide the centers O, O' harmonically, in the ratios r/r' and $-r/r'$. Those of two concentric circles coincide in the common center.*

The centers of similitude constitute a pair of opposite points of intersection of common tangents to C and C' (Ch. XVI, § 13); the external center is the point of intersection of the two common tangents which, when real, are the direct common tangents, and the internal center is the point of intersection of the two common tangents which, when real, are the transverse common tangents. When C and C' are concentric, these two pairs of tangents coincide in the isotropic lines through the common center.*

It is evident from these considerations that a center of similitude is outside, on, or inside *both* circles.

THEOREM 3. *The centers of similitude of the pairs of circles formed*

* If C, C' are tangent at a finite point P, the tangents of one pair coincide in the tangent at P and their point of intersection becomes the point P.

from three circles with real traces and noncollinear centers are the vertices of a complete quadrilateral, whose diagonal triangle is the triangle determined by the centers of the circles.

Let the circles be C_1, C_2, C_3, their centers O_1, O_2, O_3, and their radii r_1, r_2, r_3. Denote the external and internal centers of similitude of C_k and C_l by E_{kl} and I_{kl} respectively. The three external centers of similitude E_{12}, E_{23}, E_{31} divide the line-segments O_1O_2, O_2O_3, O_3O_1 in ratios r_1/r_2, r_2/r_3, r_3/r_1 whose product is unity. Hence they are collinear by the Theorem of Menelaus (Ch. VI, § 6). Similarly, the external center of similitude of each of the three pairs of circles is collinear with the internal centers of similitude of the other two pairs. Thus the six centers of similitude lie by threes on four lines. The diagonals of the complete quadrilateral formed by these lines are the sides of the triangle $O_1O_2O_3$. Why?

The four sides of the quadrilateral are known as the *axes of similitude* of the three circles, that containing the external centers of similitude as the *external axis*, and the other three as the *internal axes*.

Circle of Similitude of Two Circles. The centers of similitude, E and I, of two circles C_1, C_2 with distinct centers O_1, O_2 and unequal radii r_1, r_2 are distinct and finite. There exists, then, a circle described on EI as diameter. This circle is known as the *circle of similitude* of C_1, C_2.

When O_1, O_2 are the same point O, the circle of similitude degenerates into the null circle with center at O. When $r_1 = r_2$, it degenerates into the line at infinity and the line through I perpendicular to O_1O_2.

In each of these special cases, the circle of similitude evidently is coaxal with C_1, C_2. To prove that this is true also in the general case, it suffices to show that the ratio of the powers of one of the centers of similitude with respect to C_1, C_2 is equal to the ratio of the powers of the other (§ 5, Ex. 1). If P is either center of similitude, we have, by Th. 2,

$$\frac{O_1P^2}{r_1^2} = \frac{O_2P^2}{r_2^2},$$

whence

$$\frac{O_1P^2 - r_1^2}{r_1^2} = \frac{O_2P^2 - r_2^2}{r_2^2}$$

or

$$\frac{p_1}{r_1^2} = \frac{p_2}{r_2^2}, \tag{2}$$

where p_1, p_2 are the powers of P with respect to C_1, C_2. Thus, the proposition is proved.

Inspection shows that (2) holds for the point O of the first of the special cases and for the point I of the second. Hence we have in all cases:

THEOREM 4. *The circle of similitude of two circles with real traces is coaxal with them. It is the locus of a point which moves so that its powers with respect to the circles are as the squares of their radii.*

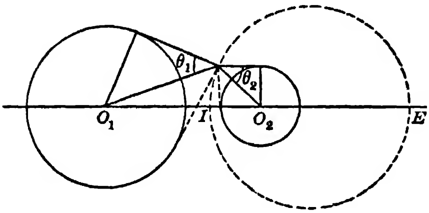


FIG. 6

The condition that a point P lie on the circle of similitude, written in terms of the squares of the tangent distances t_1^2, t_2^2 from P to C_1, C_2, is

$$\frac{t_1^2}{r_1^2} = \frac{t_2^2}{r_2^2}.$$

An equivalent condition, as is evident from Fig. 6, is

$$\cot^2 \theta_1 = \cot^2 \theta_2.$$

Thus we have obtained the property from which the circle of similitude derives its name.

THEOREM 5. *The circle of similitude is the locus of points at which the two circles subtend the same angle.**

EXERCISES

1. Show that, if two circles are tangent to a third, one of their centers of similitude is collinear with the points of contact.

2. The six lines obtained by joining the center of each of three circles with real traces and noncollinear centers to the centers of similitude of the other two are the sides of a complete quadrangle, whose diagonal points are the centers of the circles.

3. Prove that the circle of similitude of two circles with real traces is the locus of a point moving so that its distances to the centers of the two circles are as their radii.

4. Show that, if the two circles of § 5, Ex. 4 cut equal segments from the straight line, the circle on which the points of intersection of the tangents lie is the circle of similitude.

* The theorem does not hold for the first of the special cases. Why not?

5. If two circles have four real distinct common tangents, the points of intersection of these common tangents, other than the centers of similitude, lie on the circle constructed on the line-segment bounded by the centers of the given circles as diameter.

Establish the following theorems concerning *the three circles of similitude of the pairs of circles formed from three given circles* with real traces and noncollinear centers.

6. Every circle orthogonal to the three given circles is also orthogonal to the three circles of similitude.

7. The circle which goes through the centers of the three given circles is orthogonal to the three circles of similitude.

8. The three circles of similitude belong to the same pencil of circles.

8. Inversion. This is the name given to the involutory transformation carrying each of two arbitrary points which are mutually inverse in a circle into the other. The circle may be a proper circle, with or without a real trace, or a finite line circle. It is known as the *circle of inversion*, and its center, when existent, as the *center of inversion.*

For the present, we shall apply the transformation only to points of the real domain. In particular, in discussing its effect on circles, we shall restrict ourselves to circles with real traces and finite line circles, as the only circles with continua of real finite points.

Inversion in a Proper Circle. According to § 3, Th. 3, two points P, P' are mutually universe in a proper circle K, with center M and square of radius a^2, if they are collinear with M and

$$\overline{MP} \cdot \overline{MP'} = a^2. \tag{1}$$

If K has a real trace: $a^2 > 0$, then P and P' lie on the same side of M, each point on K is carried by the inversion into itself, and the points, other than M, which are inside K invert into the finite points outside K, and vice versa.

When K is without a real trace: $a^2 < 0$, P and P' lie on opposite sides of M. In this case the inversion in K may be thought of as the product of the reflection in the point M and the inversion in the circle K' with a real trace whose center is M and square of radius $|a^2|$. The points of K' invert into the points of K' in that each goes into the point diametrically opposite to it, and the points inside K', other than M, invert into the finite points outside K', and vice versa.

In both cases, a circle with M as center inverts into a second circle with M as center. If one of these circles shrinks to the point M, the

radius of the other becomes infinite. Moreover, when one of the points P, P' recedes indefinitely, say, along a straight line through M, the other approaches M as a limit. Accordingly, we recognize each point at infinity as an inverse of M, and vice versa.

THEOREM 1. *The real points of a proper circle are fixed points of the inversion in the circle. The remaining (real) finite points of the plane, with the exception of the center of inversion, are inverse in pairs. The inverse of a point at infinity is the center of inversion and inverse to the center of inversion is every point at infinity.*

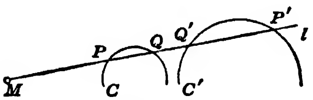


FIG. 7

What corresponds by our inversion to a circle? Consider first a circle C with a real trace which does not go through M. Let l be an arbitrary line through M, intersecting C in the points P, Q and the curve C' into which C is carried in the points P', Q' inverse to P, Q. Then

$$\overline{MP}\cdot\overline{MP'} = \overline{MQ}\cdot\overline{MQ'} = a^2.$$

Hence

$$\frac{\overline{MQ'}}{\overline{MP}} = \frac{\overline{MP'}}{\overline{MQ}} = k. \tag{2}$$

For the common value k of the two ratios, we have

$$k = \frac{\overline{MQ'}}{\overline{MP}} = \frac{\overline{MQ'}\cdot\overline{MQ}}{\overline{MP}\cdot\overline{MQ}} = \frac{a^2}{p}, \tag{3}$$

where p is the power of M with respect to C. Thus k is a constant, $\neq 0$, independent of the line l. Equations (2) then say that the homothetic transformation with M as center and k as ratio of similitude carries P into Q' and Q into P' for each line l, and hence carries C into C'. But a homothetic transformation with M as center carries a circle with a real trace not containing M into a second circle with the same properties.*

THEOREM 2 *a*. *A circle with a real trace which does not pass through the center of inversion inverts into a circle of the same description.*

Consider next a circle C with a real trace which goes through M. If P is an arbitrary point on C, other than M and the point A diametri-

* Though the figures for this, and the following, proof are drawn for the case $a^2 > 0$, the proofs themselves are valid whether $a^2 > 0$ or $a^2 < 0$.

cally opposite to M, we have

$$\overline{MP} \cdot \overline{MP'} = \overline{MA} \cdot \overline{MA'} = a^2,$$

whence

$$\frac{\overline{MP}}{\overline{MA}} = \frac{\overline{MA'}}{\overline{MP'}}.$$

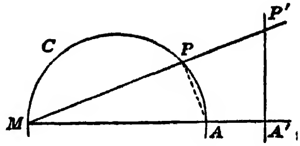


FIG. 8

The triangles MPA and $MA'P'$ are, then, similar; hence $\angle\ MA'P'$ is a right angle. Thus, the locus of P', as P takes on all positions on C, other than M, is the straight line perpendicular to the line MA at A'. On the other hand, the point M inverts into the line at infinity.

THEOREM 2 *b*. *A circle C with a real trace which goes through the center of inversion inverts into a finite line circle C' not through the center of inversion, and conversely. The finite line of C' is parallel to the tangent to C at the center of inversion.*

The converse follows from the fact that inversion is involutory.

Finally, we have:

THEOREM 2 *c*. *A finite line circle through the center of inversion inverts into itself.*

To express in simple form the general facts embodied in these and later detailed results, let us agree, for the remainder of this section, to mean by a circle, except in the case of the circle of inversion, either a circle with a real trace or a finite line circle.

THEOREM 3. *Every inversion carries a circle into a circle.*

The truth of the theorem in the case of an inversion in a finite line circle is apparent when we recall that, in this case, the inversion is simply a reflection in a finite line (§ 3, Th. 2).

The Conformal Properties of Inversion. A point transformation of the plane is *conformal* if it preserves the magnitude of every angle; in particular, *directly conformal* if it also preserves the sense, and *inversely conformal* if it reverses the sense.

It is intuitively evident that a rigid motion is directly conformal, and a reflection in a line, inversely conformal.

The definition needs to be formulated more explicitly. Suppose that a point P and two curves C_1, C_2 through P are carried into a point P' and two curves C_1', C_2' through P'. Then, to chosen directions of motion along C_1 and C_2 will correspond definite directions of motion

along C_1' and C_2'. Draw at P the tangents to C_1, C_2 directed in the chosen directions and at P' the tangents to C_1', C_2' directed in the corresponding directions. Denote the angle from the directed tangent to C_1 to the directed tangent to C_2 by $\phi : -\pi \leqq \phi \leqq \pi$, and the angle from the directed tangent to C_1' to the directed tangent to C_2' by $\phi' : -\pi \leqq \phi' \leqq \pi$. The transformation is, then, directly conformal if, always, $\phi' = \phi$, and inversely conformal if, always, $\phi' = -\phi$.

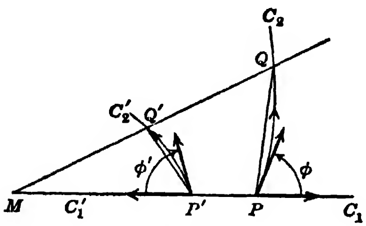


FIG. 9

The reader should now apply the amplified definition to a rigid motion and a reflection in a line.

Since reflection in a line is inversely conformal, it is reasonable to expect that inversion in a proper circle is also. In this case, let us first think of C_1 as a line through M and of C_2 as an arbitrary curve intersecting C_1 in a point P other than M (Fig. 9). Let Q be a point on C_2 in the neighborhood of P and let Q' be the inverse point on C_2'. From $\overline{MP}\cdot\overline{MP'} = \overline{MQ}\cdot\overline{MQ'}$ follows $\overline{MP'}/\overline{MQ} = \overline{MQ'}/\overline{MP}$. Hence $\triangle\, MP'Q'$ is similar to $\triangle\, MQP$ and $\angle\, MP'Q' = \angle\, MQP$. The limits of these angles, when Q approaches P as a limit, are respectively $|\phi'|$ and $|\phi|$. Therefore $|\phi'| = |\phi|$ and, since ϕ' and ϕ are clearly of opposite sign, $\phi' = -\phi$.

The case in which C_1 and C_2 are both arbitrary curves through a point P other than M is readily treated by drawing the line MP and applying the result just obtained.

THEOREM 4. *Every inversion is inversely conformal.**

COROLLARY. *Tangent curves invert into tangent curves.*

Closely related to the fact that inversion is a conformal transformation is our original definition of inverse points (§ 3). In the light of § 3, Th. 4, we may now restate this definition in a more general form.

DEFINITION. *Two distinct points are mutually inverse in a given proper or finite line circle if every circle through them is orthogonal to the given circle.*

The definition and Theorem 4 enable us to prove with ease two important properties of inversion.

* The center of a proper circle of inversion is an exceptional point.

Theorem 5. *A necessary and sufficient condition that a circle, other than the circle of inversion, invert into itself is that it be orthogonal to the circle of inversion.*

The theorem is self-evident if the inversion is a reflection in a line. Suppose, then, that the circle of inversion is a proper circle K with center M. A circle with a real trace, other than K, inverts into itself only if each two points of it which are collinear with M are mutually inverse in K, and hence, by the definition, only if it is orthogonal to K. And a finite line circle inverts into itself, by Th. 2, only when it goes through M, that is, is orthogonal to K.

Theorem 6. *Every inversion carries two points which are mutually inverse in a circle into two points which are mutually inverse in the transformed circle.**

A circle C and two points P_1, P_2 are carried by an inversion into a circle C' and two points P_1', P_2'. It is to be proved that, if P_1, P_2 are mutually inverse in C, then P_1', P_2' are mutually inverse in C'. Since P_1, P_2 are mutually inverse in C, two circles C_1, C_2 through P_1, P_2 cut C orthogonally, and so are carried into two circles C_1', C_2' through P_1', P_2' cutting C' orthogonally (Th. 4). Hence P_1', P_2' are mutually inverse points in C'.

An interesting consequence of this theorem is that an inversion in a proper circle carries two points which are reflections in a line, not through the center of inversion, into two points which are mutually inverse in a proper circle. In other words, an inversion transforms a reflection in a line, in general, into an inversion in a proper circle.

EXERCISES

1. Show that the equations of the inversion in a proper circle with center M and square of radius a^2 are

$$x' = \frac{a^2 x}{x^2 + y^2}, \qquad y' = \frac{a^2 y}{x^2 + y^2},$$

where (x, y) are rectangular coordinates referred to M as origin.

2. Prove Theorem 2 analytically.

3. Deduce Theorem 5 directly from Theorem 4 in the case that the circle of inversion has a real trace.

4. Show that a pencil of circles inverts into a pencil of circles, and that it inverts into itself in case the circle of inversion belongs to it.

* The reader will recall that by "points" we always mean real *finite* points unless the contrary is explicitly stated.

5. A variable circle moves so that it always cuts a given proper circle orthogonally and remains tangent to a fixed circle. Show that it is always tangent to a second fixed circle, which is, in general, distinct from the first.

6. A straight line L and a finite point P, not on L, are given. The isotropics through P intersect L in P_1, P_2 and the second isotropics through P_1, P_2 meet in P'. Show that P' is the reflection of P in L.

7. Prove that, if a proper circle is substituted for the line L in the preceding exercise, the point P' is the inverse of P in the circle. See Ch. VIII, § 4.

9. Circles of Antisimilitude of Two Circles. In § 7, we determined the homothetic transformations which carry a given circle with a real trace into a second given circle with a real trace. We now seek the inversions which carry the one circle into the other.

The proof of Theorem 2 a, § 8, tells us that, if M is the center of a proper circle K in which the given circles C, C' are mutually inverse, there exists a homothetic transformation with M as center which carries C into C'. Hence M is a center of similitude of C and C'.

Except in two special cases which we shall discuss presently, the steps of the proof in question can be reversed. Thus, if M is a center of similitude of C, C', there exists a proper circle K with center at M in which C and C' are mutually inverse. The square of the radius, a^2, of this circle is, by § 8, (3),

$$a^2 = kp, \tag{1}$$

where k is the ratio of similitude of the homothetic transformation with M as center which carries C into C', and p is the power of M with respect to C.

THEOREM 1. *There are, in general, two and only two proper circles in which two given circles with real traces are mutually inverse. Their centers lie respectively in the centers of similitude of the given circles.*

The two circles are known as the *circles of antisimilitude* of the given circles, that with its center in the external center of similitude as the *external*, and the other as the *internal*, circle of antisimilitude.

There are two exceptions to Theorem 1, one of which is removable. If C, C' are tangent and M is the point of contact, p is zero and the circle K becomes the null circle with center at M; there is, then, no inversion in K. When C, C' have the same radius, and M is their infinite center of similitude, K is to be replaced by the finite line circle whose finite line is the radical axis of C, C'; this is evidently the only case in which C and C' are reflections in a line.

We agree to look upon the null circle and finite line circle of these two cases as degenerate circles of antisimilitude. *There are then always two distinct circles of antisimilitude.*

If M is the external center of similitude, $k > 0$ and hence (1) tells us that $a^2 > 0$ only when $p > 0$. On the other hand, if M is the internal center of similitude, $k < 0$ and hence $a^2 > 0$ only when $p < 0$.

THEOREM 2. *The external (internal) circle of antisimilitude has a real trace if and only if the external (internal) center of similitude is outside (inside) the given circles.*

In particular, both circles of antisimilitude of two intersecting circles have real traces; in the case of two tangent circles, one has a real trace and the other is a null circle; for two nonintersecting circles, the external or the internal one has a real trace according as the given circles are mutually external or one surrounds the other.

THEOREM 3. *There is always at least one circle with a real trace* * *in which two given circles with real traces are mutually inverse.*

In each of the special cases, the degenerate circle of antisimilitude is coaxal with the given circles. This is true also in the general case. For example, if the given circles are tangent, the circle in which they are mutually inverse must be tangent to them at their common point.

THEOREM 4. *The two circles of antisimilitude are coaxal with the given circles.*

Let us now consider the six circles of antisimilitude of the pairs of circles formed from three given circles with noncollinear centers. The centers of these circles, since they are the centers of similitude of the pairs of given circles, lie by threes on the four axes of similitude of the given circles. Three circles with centers on the same axis, for example, the three external circles of antisimilitude, are coaxal. This is a consequence of § 6, Th. 9, inasmuch as we can exhibit two circles orthogonal to all three, namely, the finite line circle whose finite line is the line of centers and the circle orthogonal to the three given circles. The latter circle is, in fact, orthogonal to all six circles of antisimilitude, since it follows from Theorem 4 that every circle orthogonal to two circles is orthogonal to their two circles of antisimilitude.

THEOREM 5. *The six circles of antisimilitude of the pairs of circles formed from three circles with real traces and noncollinear centers belong*

* We have included the finite line circle of the second special case among the circles with real traces, both here and in the preceding paragraph.

by threes to four coaxal systems. The lines of centers of the four systems are the axes of similitude of the given circles.

EXERCISES

1. Prove that each of the circles of antisimilitude of two intersecting circles makes equal angles with them.

2. Show that the two circles of antisimilitude are always mutually orthogonal.

3. Show that, if two circles C, C' with real traces, with centers O, O' and radii r, r', are mutually inverse in a circle with center M and square of radius a^2, then

$$\frac{r'}{r} = \left|\frac{a^2}{p}\right| = \frac{MO'}{MO}, \qquad pp' = a^4,$$

where p and p' are the powers of M with respect to C and C'.

4. An inversion carries a circle of antisimilitude of two circles into a circle of antisimilitude of the transformed circles.

5. Two circles with real traces are carried into two circles with equal radii by an inversion with center on a circle of antisimilitude, but not on the given circles.

6. When can three circles with real traces be inverted into three circles with the same radius?

10. Circles Cutting Two or Three Circles at the Same Angles. There are two angles θ, $0 \leqq \theta \leqq \pi$, between two circles with real traces which intersect or are tangent. They are given by

$$\cos^2\theta = \frac{(r_1^2 + r_2^2 - \overline{O_1O_2}^2)^2}{4\, r_1^2 r_2^2},$$

or in terms of the coefficients a_1, a_2, a_3, a_4 and b_1, b_2, b_3, b_4 in the equations of the circles, by

$$(1) \qquad \cos^2\theta = \frac{(a, b)^2}{(a, a)(b, b)},$$

where the symbols (a, a), (a, b) are those introduced in §§ 1, 6. The angles between any two circles a, b for which $(a, a)(b, b) \neq 0$, we agree to define by the latter formula.*

THEOREM 1. *Two distinct circles make definite angles with one another provided neither is a null circle or the infinite line circle.*

Circles Cutting Two Circles at the Same Angles. We shall say that a

* As often as not, the angles thus defined are imaginary, so that the restriction $0 \leqq \theta \leqq \pi$ is inapplicable. In these cases we should think of (1) as defining the cosines of the angles rather than the angles themselves. We shall however, for the sake of simplicity, always talk of the angles.

circle C *cuts** two given circles C_1, C_2 with real traces at the same angles, if it makes definite angles with C_1 and C_2, and the two angles which it makes with C_1 are equal to the two angles which it makes with C_2. It is necessarily a proper or a finite line circle, by Th. 1.

Circles which actually *intersect* the two given circles C_1, C_2 at the same angles are readily obtained by considering the inversions which interchange C_1 and C_2. An inversion which carries C_1 into C_2 will carry a circle C which intersects C_1 at certain angles into a circle which intersects C_2 at the same angles. Accordingly, if C inverts into itself, it will intersect C_1 and C_2 at the same angles. But C is its own inverse if it is orthogonal to the circle of inversion, and the circle of inversion is a circle of antisimilitude of C_1, C_2.

Of the circles orthogonal to a circle of antisimilitude, there are evidently ∞^2 proper or finite line circles which intersect C_1 and hence also intersect C_2. We have proved that these ∞^2 circles cut C_1, C_2 at the same angles.†

We now consider the problem analytically. Let a, b be the given circles. For an arbitrary proper or finite line circle s cutting them at the same angles, we have

$$\frac{(a,\,s)^2}{(a,\,a)(s,\,s)} = \frac{(b,\,s)^2}{(b,\,b)(s,\,s)},$$

or

$$\sqrt{b,\,b}\,(a,\,s) \mp \sqrt{a,\,a}\,(b,\,s) = 0.$$

Hence, a necessary and sufficient condition that s cut a, b at the same angles is that s be orthogonal to one or the other of the circles

$$(2) \qquad \sqrt{b,\,b}\,a - \sqrt{a,\,a}\,b, \qquad \sqrt{b,\,b}\,a + \sqrt{a,\,a}\,b.$$

These circles are the circles of antisimilitude of a, b. For, we have already exhibited ∞^2 circles with the prescribed property which are orthogonal to a circle of antisimilitude of a and b, and ∞^2 circles cannot be orthogonal to two distinct circles.

THEOREM 2. *The proper and finite line circles which are orthogonal to one or the other of the two circles of antisimilitude of two given circles with real traces cut the given circles at the same angles and are the only circles with this property.*

* We use "cuts" rather than "intersects" since "intersects" means for us "intersects in real points."

† The argument breaks down if C_1, C_2 are tangent circles and the circle of antisimilitude is the null circle with center in the point of contact. It is, however, geometrically evident that the conclusion still remains valid.

Let us call these circles the circles K, in particular, those orthogonal to the external circle of antisimilitude, the circles K_e, and those orthogonal to the internal circle of antisimilitude, the circles K_i.

COROLLARY. *The circles common to the two sets of circles K_e, K_i are the circles which cut the given circles orthogonally.*

Thus far we have been interested merely in having the two angles at which K cuts C_1 the same as those at which K cuts C_2. We shall now adopt a rule for singling out one of the two angles between two circles as *the* angle at which they cut. The circles K will then fall into two groups, those for which the angle at which K cuts C_2 is equal to the angle at which K cuts C_1, and those for which the angle at which K cuts C_2 is the supplement of the angle at which K cuts C_1. These two groups will have in common the same circles as do the sets K_e, K_i and will, in fact, be identical with these sets.

We define *the angle between two circles* for two intersecting or tangent circles with real traces. Let each of the two circles be thought of as traced in the positive, that is, counterclockwise, sense. Direct the tangents to them at a common point in the senses which at the point coincide with those on the circles. The angle ϕ, $0 \leqq \phi \leqq \pi$, between the directed tangents is the same at each point common to the two circles and is defined as *the* angle at which the circles intersect. It is readily seen that

$$\cos \phi = \frac{r_1^2 + r_2^2 - \overline{O_1O_2}^2}{2\, r_1 r_2}.$$

Hence, if s is a circle with a real trace intersecting the given circles a, b, the angle at which it intersects a is equal to the angle at which it intersects b if and only if

$$\frac{(a,\, s)}{\sqrt{a,\, a}\,\sqrt{s,\, s}} = \frac{(b,\, s)}{\sqrt{b,\, b}\,\sqrt{s,\, s}},$$

provided that $a_1b_1 > 0$.* Extending the validity of this criterion in the usual way, we find that a proper or a finite line circle s cuts a, b

* The condition appears first in the form

$$\frac{(a,\, s)}{\sqrt{a,\, a}\sqrt{s,\, s}}\,\frac{|a_1|\cdot|s_1|}{a_1 s_1} = \frac{(b,\, s)}{\sqrt{b,\, b}\sqrt{s,\, s}}\,\frac{|b_1|\cdot|s_1|}{b_1 s_1}.$$

Hence the necessity of the assumption: $a_1b_1 > 0$. Since $a_1b_1 \neq 0$ and the a's and b's are homogeneous parameters, the assumption can always be fulfilled.

at the same angle or at supplementary angles according as

(3) $\sqrt{b, b}\,(a, s) - \sqrt{a, a}\,(b, s) = 0$ or $\sqrt{b, b}\,(a, s) + \sqrt{a, a}\,(b, s) = 0,$

that is, according as it is orthogonal to the first or the second of the circles of antisimilitude (2).

The first of the circles (2) is the external, and the second the internal, circle of antisimilitude. To establish this fact, it suffices to prove that it is true in one case. For, the general truth of it will then follow by continuity, inasmuch as the two circles of antisimilitude never coincide. In case a, b are intersecting circles, it is evident that the internal circle of antisimilitude intersects a, b at the same angle (§ 9, Ex. 1; Fig. 10). Thus the internal circle of antisimilitude is a circle orthogonal to the first of the circles (2). The first of the circles (2) must, therefore, be the external circle of antisimilitude.

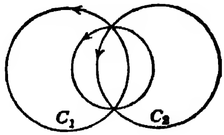


FIG. 10

THEOREM 3. *The circles which cut the given circles at the same angle are the circles K_e orthogonal to the external circle of antisimilitude. Those cutting them at supplementary angles are the circles K_i orthogonal to the internal circle of antisimilitude.*

Circles Tangent to Two Circles. According to (1), two circles a, b, neither of which is a null circle or the infinite line circle, make with one another the angles 0 and π if and only if

$$(a, a)(b, b) - (a, b)^2 = 0. \tag{4}$$

This, then, is the condition that the two circles be mutually tangent.

Two circles with real traces may be tangent internally or tangent externally. In the first case, the angle ϕ at which they cut is 0; in the second case, $\phi = \pi$. Accordingly, we shall say that a proper or finite line circle which is tangent to our two given circles C_1, C_2 has the same or opposite types of contact with C_1 and C_2 according as it cuts C_1, C_2 at the same angle or at supplementary angles. Hence:

THEOREM 4. *Of the proper and finite line circles tangent to two given circles C_1, C_2 with real traces, those which have contacts of the same type with C_1, C_2 are orthogonal to the external circle of antisimilitude, and those which have contacts of opposite types with C_1, C_2 are orthogonal to the internal circle of antisimilitude.*

If b is a null circle whose center lies on the circle a, then $(b, b) = 0$,

$(a, b) = 0$, and (4) is satisfied: the two circles are tangent. We add, then, to the circles of Theorem 4 whatever null circles there may be whose centers are common to the given circles C_1, C_2. The contacts of these null circles with C_1, C_2 are indeterminate in type inasmuch as both equations (3) are satisfied when $(a, s) = 0$ and $(b, s) = 0$.*

Circles Cutting Three Circles at the Same Angles. Let C_1, C_2, C_3 be three circles with real traces and noncollinear centers. The circles, each of which cuts all three at the same angle, are, by Th. 3, the circles which are orthogonal to the three external circles of antisimilitude of the pairs of circles formed from C_1, C_2, C_3. These three circles of antisimilitude belong to a pencil (§ 9, Th. 5). The circles sought are therefore the circles of the conjugate pencil. The line of centers of this pencil is the perpendicular dropped from the radical center of C_1, C_2, C_3 on the external axis of similitude. For, the line circle with this axis as finite line and the circle which has the radical center as center and is orthogonal to C_1, C_2, C_3 are two circles which cut C_1, C_2, C_3 at the same angle.†

A similar argument applies to the circles each of which cuts two of the given circles at the same angle and the third at the supplementary angle.

THEOREM 5. *The circles, each of which cuts three given circles C_1, C_2, C_3 with real traces and noncollinear centers at the same or supplementary angles, are the circles of the four coaxal systems which are conjugate respectively to the coaxal systems determined by the six circles of antisimilitude of the three given circles. Each of the circles of the coaxal system S_0 conjugate to that determined by the external circles of antisimilitude cuts the given circles at the same angle. Each of the circles of the coaxal system S_k*

* A null circle whose center lies on a given proper or line circle is not only tangent to the given circle, but also, by § 6, Th. 4, orthogonal to it. The explanation of this paradox is simple. The null circle consists of the two isotropic lines through its center O. If we were considering it by itself, we should recognize as tangent lines to it at O only these isotropics. However, in finding it both tangent and orthogonal to the given circle through O, we have recognized also as tangent to it at O the lines through O which are respectively coincident with, and perpendicular to, the tangent at O to the given circle. To be consistent, we should go further and recognize every line through O as tangent to it. It makes then every angle with the given circle.

Accordingly, we add the null circles whose centers are common to the given circles, not only to the circles of Theorem 4, but also to those of Theorems 2, 3.

† If C_1, C_2, C_3 have equal radii, the external axis of similitude is the line at infinity, and the pencil in question consists of the circles with the radical center as center.

conjugate to that determined by the external circle of antisimilitude of C_i, C_j *and the internal circles of antisimilitude of* C_i, C_k *and* C_j, C_k *cuts* C_i *and* C_j *at the same angle and* C_k *at the supplementary angle. The lines of centers of the four coaxal systems are the perpendiculars dropped from the radical center on the four axes of similitude.**

Circles Tangent to Three Circles. The circles of a coaxal system of Theorem 5 which are tangent to one of the given circles C_1, C_2, C_3 are tangent to all three.

LEMMA. *There are just two circles in a pencil of circles which are tangent to a given circle with a real trace not belonging to the pencil. They may be real and distinct, real and coincident, or conjugate-imaginary.*

By means of this Lemma, the proof of which we leave to the reader, we conclude

THEOREM 6. *There are eight circles tangent to three given circles* C_1, C_2, C_3 *with real traces and noncollinear centers. They consist of a pair of circles from each of the coaxal systems described in Theorem 5. Each circle of the pair from the system* S_0 *has contacts of the same type with the given circles, and each circle of the pair from the system* S_k *has contacts of the same type with* C_i, C_j *and contact of the opposite type with* C_k. *The two circles of a pair are real and distinct, real and coincident, or conjugate-imaginary.*

We return now to the Lemma. Let C denote the given circle, K_1, K_2 the two circles of the pencil which are tangent to C, and K a circle of the pencil which is orthogonal to C (§ 6, Ex. 6). Apply the inversion in K, assuming that K is not a null circle. The circle C inverts into itself and, since the pencil inverts into itself (§ 8, Ex. 4), the circles K_1, K_2 are carried into themselves. If each of these circles inverts into itself, they are both orthogonal to K and hence coincide. Otherwise they are distinct and each inverts into the other. In either case K_1, K_2 are mutually inverse in K.†

To apply the result to the problem in hand, we note that the circle orthogonal to C_1, C_2, C_3 plays the rôle of the circle K for each of the four coaxal systems.

* A null circle of one of the systems S is to be included only when its center lies on all three circles, and the infinite line circle is never to be included. The subscripts i, j, k represent a cyclic order of the numbers 1, 2, 3.

† Since we have considered inversion only for the real domain, the proof applies only to the case when the circles K_1, K_2 are real.

Corollary. *The circles of each of the four pairs of circles are mutually inverse in the circle orthogonal to the three given circles, provided this circle is not a null circle.*

If this circle is a null circle, the given circles all go through its center. The null circle is tangent to each of the given circles and counts at least four times as a solution of our problem, inasmuch as it belongs to each of the four coaxal systems. This is the only case in which two or more of the four pairs of circles have a circle in common. Why?

EXERCISES

1. Prove that there are in general two circles K of Theorem 2 with a given point as center, a circle K_e and a circle K_i. What are the exceptions?

2. Prove that, if a proper circle intersects two circles at the same *angles*, the four points of intersection fall into two pairs collinear with one center of similitude. When is this the external, and when the internal, center of similitude?

3. What is the analogous theorem concerning a circle tangent to two circles?

4. Show that, if a circle intersects two given circles with real traces orthogonally, in four points no three of which are collinear, two of the diagonal points of the complete quadrangle with vertices in the four points are the centers of similitude of the given circles. Where is the third diagonal point?

5. Two pairs of circles with real traces are so situated that the two circles of each pair cut those of the other pair at the same *angle*. Show that the external center of similitude of each pair lies on the radical axis of the other pair.

6. Show directly that the circles of antisimilitude of the circles a, b are the circles (2).

7. Prove that either all the circles of a pencil or at most two cut two given circles with real traces at the same *angles*. When does the first case occur? Why is the qualification "at most" necessary?

8. Establish the Lemma.

B. The Geometry of Inversion

In Part A, we discussed the circle in the Cartesian plane from the point of view of the group of rigid motions. We shall now consider it in relation to a larger group in which the transformation of inversion plays the fundamental rôle. In the beginning we shall restrict ourselves exclusively to the real plane and shall understand by "inversion" an inversion in a circle with a real trace.

11. The Real Plane of Inversion. In our preliminary study of inversion (§ 8), we learned that to the center of inversion corresponds

every ideal point. Accordingly, if there are ∞^1 ideal points, as in the Cartesian plane, inversion fails of being one-to-one. However, if there is created only one ideal point, this blemish is removed.

AGREEMENT 1. *A single ideal point is created, to be known as the point at infinity and to correspond in every inversion to the center of inversion.*

In the real plane of inversion, that is, the real finite plane with *the* point at infinity adjoined, every inversion is one-to-one without exception.

By a point we shall now mean *any* point of the real plane of inversion.

Circles. Since we are restricting ourselves to the real domain and have replaced the usual line at infinity by a single point at infinity, the circles of Part A become:

(*a*) a proper circle: always a circle with a real trace;

(*b*) a finite line circle: a straight line; *

(*c*) a null circle or the infinite line circle: a point.

For the present, it will be convenient to treat a null circle as the point that it actually is and to mean by a *circle* simply *a proper circle or a straight line.*

The straight lines are now the inverses of the circles through the center of inversion (§ 8, Th. 2 *b*, *c*). Logical consistency demands, then, that every straight line contain the inverse of the center of inversion, that is, the ideal point.

AGREEMENT 2. *The point at infinity lies on every straight line and on no other circle.*

The straight lines are thus the circles which pass through the point at infinity. Three distinct points determine a unique circle; if one is the point at infinity, the circle is the straight line which joins the other two.

Let us recall the fundamental properties of inversion.

THEOREM 1. *Every inversion* (*a*) *carries a circle into a circle,* (*b*) *is inversely conformal, and* (*c*) *carries two points which are mutually inverse in a circle into two points which are mutually inverse in the transformed circle.*

* Since a finite line circle of Part A consists of a finite straight line and the line at infinity, it appears here first as a straight line and the point at infinity. The point at infinity is, however, absorbed by the straight line; see Agreement 2.

By § 8, Th. 2 *b*, two parallel lines invert into two circles which are mutually tangent at the center of inversion, and conversely. Accordingly, we make

AGREEMENT 3. *Two parallel lines are mutually tangent at the point at infinity.*

It is evident that in the real plane of inversion two distinct circles have no, one, or two points in common. We shall describe them in the respective cases as nonintersecting, tangent, or intersecting. In particular, two straight lines are tangent or intersecting.

EXERCISES

1. Show that the pencils of circles of types (*a*), (*b*), (*c*) of Fig. 3 are now special cases of the pencils of circles of types (*A*), (*B*), (*C*) respectively.

2. Prove that, if A, B, C are distinct collinear points, there is a unique proper or line circle through C in which A and B are mutually inverse. Give a construction for this circle.

3. The criterion to the effect that two distinct points are mutually inverse in a given circle if and only if every circle through them is orthogonal to the given circle was restricted, in Part A, by the demand that both points be finite. Show that it is also valid, now, when one of the points is *the* point at infinity. Hence prove that property (*c*) of Theorem 1, which has thus far been established only in the finite plane, actually holds in the (extended) plane of inversion.

12. Possibilities and Applications of Inversion.

THEOREM 1. *A proper circle may be inverted into a straight line.*

It suffices to take the center of inversion on the proper circle.

THEOREM 2. *A circle and a point not on it may be inverted into a proper circle and its center.*

Let the inverse of the given point P, in the given circle C, be O. Invert in a circle with O as center.* Since the points P and O are mutually inverse in C, the transformed points, P' and the point at infinity, are mutually inverse in the transformed circle C'. Hence C' is a proper circle with P' as its center.

THEOREM 3. *Two intersecting circles may be inverted into two intersecting straight lines, two tangent circles into two parallel lines, and two nonintersecting circles into two concentric proper circles.*

The proof is left to the reader.

* O may be the point at infinity. What then?

Applications. Inversion may be employed, in much the same way as polar reciprocation and projection, both to obtain generalizations of special theorems and to prove general propositions by reducing them to special cases. As an example of the latter application, we establish again the important fact:

THEOREM 4. *There exist two distinct points mutually inverse in each of two circles if and only if the circles are nonintersecting.**

Since two circles can always be inverted into a proper circle and a straight line, it suffices to prove the theorem for this special case. Two points P_1, P_2 which are to be mutually inverse in both a proper circle and a line must lie on the perpendicular to the line which passes through the center O of the circle, and have their directed distances x_1, x_2 from O connected by the relations

$$x_1x_2 = a^2, \qquad x_1 + x_2 = 2\,d,$$

where d is the directed distance from O to the line and a is the radius of the circle. These relations say that x_1, x_2 are the roots of the quadratic equation

$$x^2 - 2\,dx + a^2 = 0.$$

But this equation has real, distinct roots if and only if $d^2 > a^2$, that is, when and only when the circle and the line are nonintersecting.

EXERCISES

1. Establish Theorem 3.

2. Show in detail how inversion may be employed to obtain from the special pencils of circles (a), (b), (c) of Fig. 3 the general pencils (A), (B), (C).

3. The same in the case of the special and general orthogonal systems $(a\ c)$, $(b\ b)$ and $(A\ C)$, $(B\ B)$ of Fig. 4.

4. There is a unique circle through a chosen one of three distinct points in which the other two are mutually inverse.

5. If P_1, P_2, P_3 are distinct points, the circles C_1, C_2, C_3, where C_i is the circle through P_i in which P_j, P_k are mutually inverse, go through the same two points. These two points are mutually inverse in the circle through P_1, P_2, P_3, and each two of the circles C_i are mutually inverse in the third.

6. Two proper circles are tangent at a point O. A variable secant is drawn through O meeting the circles again in P_1, P_2. Find the locus of the harmonic conjugate of O with respect to P_1, P_2.

Generalize by inversion:

7. The sum of the angles of a triangle is two right angles.

* See § 4, Th. 3. Why is it not necessary to demand here that, when the circles are proper, they have distinct centers?

8. The altitudes of a triangle are concurrent.

9. A necessary and sufficient condition that a quadrilateral may be inscribed in a proper circle is that two opposite angles be equal or supplementary. Take the center of inversion in a vertex of the quadrilateral.

10. The preceding exercise. Take the center of inversion in general position.

11. Show that the problem of finding two points mutually inverse in each of two circles is identical with the problem of finding a pair of points which separates each of two given collinear pairs of points harmonically. Hence prove Theorem 4.

13. Inversion of Regions. A proper circle separates the points of the plane, except for those on it, into two sets or regions: the *interior*, containing the center, and the *exterior*. Since by inversion in the circle the interior is carried into the exterior, the point at infinity must be considered an exterior point.

Suppose that C_1 and C_2 are two proper circles which are mutually inverse in a proper circle. The two regions bounded by C_1 must invert into the two regions bounded by C_2, for inversion is a continuous transformation. But which inverts into which? Surely the interior of C_1 does not always invert into the interior of C_2.

It is evident that the fate of a single point in a region determines the fate of the region. For example, if a point outside C_1 is carried into a point inside C_2, then the exterior of C_1 inverts into the interior of C_2.

Consider the center of inversion O. Its inverse, the point at infinity, is outside both circles. Hence, if O is outside both circles, exterior inverts into exterior and interior into interior, and if O is inside both circles, the interior of each inverts into the exterior of the other.*

THEOREM 1. *If two proper circles are mutually inverse, the interior of one inverts into the interior or the exterior of the other according as the center of inversion is outside or inside the two circles.*

THEOREM 2. *If a straight line and a proper circle are mutually inverse, that one of the half-planes bounded by the line which does not contain the center of inversion inverts into the interior of the circle.*

We leave the proof of Theorem 2 to the reader, with a warning: the point at infinity lies in neither half-plane, but on the line.

As an application of these considerations we prove the following theorem.

* Why cannot O lie inside one circle and outside the other?

THEOREM 3. *A necessary and sufficient condition that there exist a circle which is orthogonal to a given circle C and in which two given points P_1, P_2 are mutually inverse is that P_1, P_2 be both on the same side of C or both on C.*

Of the points P_1, P_2, at least one is finite. Assume that P_2 is finite, and suppose, first, that it does not lie on C. An inversion with P_2 as center carries C into a proper circle C', P_1 into a point P_1', and P_2 into the point at infinity. Moreover, it carries a circle in which P_1, P_2 are inverse into a circle in which P_1' and the point at infinity are inverse, that is, into a proper circle with P_1' as center. But there exists a proper circle with P_1' as center orthogonal to C' only if P_1' is outside C', that is, on the same side of C' as the point at infinity. Hence, there exists a circle orthogonal to C in which P_1, P_2 are mutually inverse only when P_1 is on the same side of C as P_2.

If P_2 lies on C, the circle C' into which our inversion carries C is a straight line. In this case there exists a proper circle with P_1' as center orthogonal to C' only when P_1' is on C'. Hence, there exists a circle orthogonal to C in which P_1, P_2 are mutually inverse only if P_1, as well as P_2, lies on C.

It is evident that our theorem is the generalization of the fact that there exists a proper circle with a given point P_1' as center orthogonal to a given circle C' when and only when P_1' lies outside C' if C' is a proper circle, or lies on C' if C' is a straight line.

EXERCISES

1. Prove Theorem 2.

2. Show that two nonintersecting circles can be inverted into: (*a*) two proper circles which are external to one another; (*b*) two proper circles of which the first surrounds the second; (*c*) two proper circles of which the second surrounds the first.

3. Prove that the type of tangency of two mutually tangent proper circles is preserved by inversion if the center of inversion is inside both or outside both, and that it is reversed if the center of inversion lies inside one and outside the other.

4. Show that the facts stated in Ex. 3 are true also of any two tangent circles if we understand by the inside of a straight line that half-plane bounded by the line which does not contain the center of inversion, and call two tangent circles internally or externally tangent according as their interiors have or have not points in common.

5. Prove that there exists a circle which is orthogonal to two given circles and in which two given points are mutually inverse if and only if the given points

lie on a circle linearly dependent on the given circles and are both on the same side of, or both on, one of these circles.

6. Obtain a condition, necessary and sufficient that there exist a circle orthogonal to three linearly independent circles, which is similar in nature to those of Theorem 3 and Ex. 5.

14. Tangent Circles. Theorem 4 of § 10 may be restated in part as follows:

THEOREM 1. *There are two families of circles tangent to two proper circles. Each circle of the one family has contacts of the same type with the given circles, and each circle of the other family has contacts of opposite types with the given circles.*

The theorem may be established by inversion. Suppose, for example, that the given circles are nonintersecting. They may, then, be inverted into two concentric circles, and for two concentric circles the proposition is obvious. Hence, reversing the inversion completes the proof. The details of this process are left to the reader. He will find, if he uses the proper care in inverting contacts (§ 13, Ex. 3), that a family of circles tangent to two proper circles inverts always into a family of circles tangent to the two transformed circles, though not necessarily into the family of the same description.

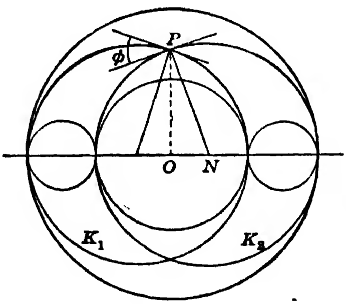


FIG. 11

THEOREM 2. *There are four circles which are tangent to two nonintersecting proper circles and cut orthogonally a chosen circle orthogonal to the two. Of the four, only two are intersecting. These two belong to the same family of circles tangent to the given two, and the angles at which they intersect are independent of the orthogonal circle chosen.*

If the two circles are concentric, a circle orthogonal to them is a straight line through their common center. The proposition is clear, in this case, from Fig. 11. Its validity in all cases follows immediately.

Poristic Systems. Let two nonintersecting proper circles be given. Under what conditions will there be a closed series of circles all tangent to the given two and each tangent to its two neighbors in the series?

It is evident that, in the case of two concentric circles C_1, C_2 (Fig. 12),

the series will close after one circuit if and only if the angle θ which an arbitrary one of its members subtends at O goes into 2π an integral number of times. This condition can be put into a form which is more readily generalized by inversion. If r_1, r_2 are the radii of C_1, $C_2 (r_1 > r_2)$, the common radius MA of the circles of the series is $\frac{1}{2}(r_1 - r_2)$ and the common distance OM of their centers from O is $\frac{1}{2}(r_1 + r_2)$, so that

$$\sin \tfrac{1}{2}\theta = \frac{r_1 - r_2}{r_1 + r_2}$$

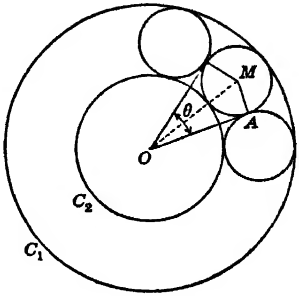


FIG. 12

Consider, on the other hand, the circles K_1, K_2 of Fig. 11. Their common radius NP is $\frac{1}{2}(r_1 + r_2)$ and the common distance ON of their centers from O is $\frac{1}{2}(r_1 - r_2)$. Hence, if ϕ is that one of the angles between them which is marked in the figure,

$$\sin \tfrac{1}{2}\phi = \frac{r_1 - r_2}{r_1 + r_2}$$

Thus, $\theta = \phi$ and the original condition for closure after one circuit, or better, the condition which guarantees closure for the *first* time after m circuits, becomes

$$\phi = \frac{2m\pi}{n},$$

where the integers m and n are relatively prime.*

How the condition carries over to the general case is clear once we realize that K_1, K_2 constitute a special instance of the two circles emphasized in Theorem 2.

THEOREM 3. *A necessary and sufficient condition that there exist a closed series of circles all tangent to two nonintersecting proper circles C_1, C_2 and each tangent to its neighbors in the series is that a certain one of the angles, formed by the two intersecting circles K_1, K_2 which are tangent to C_1, C_2 and orthogonal to a circle orthogonal to C_1, C_2, be a rational multiple m/n of 2π. The series consists of circles belonging to that family of circles tangent to C_1, C_2 which does not contain K_1, K_2 and, if m/n is in its lowest terms, it closes for the first time after m circuits and contains, then, n circles.*

* If m and n had, say, the common factor 2, we should have closure in $m/2$ circuits.

The simplest choice of the circle orthogonal to C_1, C_2 is the straight line containing their centers. Which of the angles between K_1, K_2 is to be taken is readily determined by visualizing the inversion of C_1, C_2 into two concentric circles.

EXERCISES

1. Complete the proof of Theorem 1 by inversion when one of the given circles surrounds the other. Show that the circles of one of the two families are orthogonal to a circle of antisimilitude of the given circles.

2. The same when the given circles are mutually external.

3. By inversion, show that Theorem 1 is true for two intersecting circles, and that, in this case, the circles of each family are orthogonal to a circle of antisimilitude of the given circles.

4. How many real distinct circles are there tangent to three mutually external proper circles? Substantiate your answer.

5. Let C_1 and C_2 be circles tangent at P, and let Q be a point not on either C_1 or C_2. If a circle is drawn through Q and P meeting C_1 again in P_1 and C_2 again in P_2, the circles through Q tangent respectively to C_1 and C_2 at P_1 and P_2 are tangent to one another.

6. A closed ring of four proper circles, each of which is tangent to its two neighbors, is given. Each two points of contact are assumed distinct. Show that, if the four contacts are all external, the four points of contact are concyclic.

7. Prove that, in the preceding exercise, the four points of contact are concyclic if and only if an even number of the four contacts are external or an even number internal.

15. Circular Transformations. By these we shall mean the point transformations of the plane of inversion which are one-to-one and carry a circle always into a circle. Let us collect all those we know and try to form from them the largest possible group.

We have, besides the inversions, of which that in the unit circle,

$$x' = \frac{x}{x^2 + y^2}, \qquad y' = \frac{y}{x^2 + y^2},$$

is typical, the transformations of similarity,

$$(1) \qquad \begin{aligned} x' &= r(x \cos \phi - y \sin \phi) + b_1, \\ y' &= r(x \sin \phi + y \cos \phi) + b_2, \end{aligned} \qquad r > 0.$$

The inversions do not in themselves form a group, for an inversion is inversely conformal and the product of two inversely conformal transformations is directly conformal.

On the other hand, the transformations of similarity (1) are all

directly conformal and do form a group. There must, however, be a larger group of directly conformal circular transformations in which the process of inversion is represented. To obtain this group, we first replace the typical inversion, in the unit circle, by a directly conformal circular transformation closely allied to it, namely by the transformation,

$$(2) \qquad x' = \frac{x}{x^2 + y^2}, \qquad y' = -\frac{y}{x^2 + y^2},$$

obtained by forming the product of it and the reflection $x' = x$, $y' = -y$ in the axis of x.

We now have, in (1) and (2), directly conformal circular transformations of different types. By forming products of them, we shall obtain more transformations with the same properties and eventually an entire group.

To facilitate the process of forming the products in question, we note that the equations (1), since x, y, r, ϕ, b_1, b_2 are real, are equivalent to the single equation

$$x' + i\,y' = r(\cos\phi + i\sin\phi)(x + i\,y) + (b_1 + i\,b_2),$$

and hence to

$$(1\,a) \qquad z' = \alpha z + \beta, \qquad \alpha \neq 0,$$

where

$$z' = x' + i\,y', \qquad z = x + i\,y,$$

and $\alpha = r(\cos\phi + i\sin\phi)$, $\beta = b_1 + i\,b_2$. Similarly, the equations (2) are equivalent to

$$(2\,a) \qquad z' = \frac{1}{z}.$$

Finally, we remark that the product of transformations (1) and (2) may be obtained by the usual process applied to equations (1 a) and (2 a).

Since (1 a) and (2 a) are linear transformations of z into z', the product of any finite number of them will be a linear transformation of z into z', that is, a transformation of the form

$$(3) \qquad z' = \frac{\alpha z + \beta}{\gamma z + \delta}, \qquad \alpha\delta - \beta\gamma \neq 0,$$

where $\alpha, \beta, \gamma, \delta$ are complex numbers.

Conversely, every transformation (3) is a transformation (1 a) or (2 a), or a product of transformations of these types. If $\gamma = 0$, (3) is

of type (1 a). If $\gamma \neq 0$ and we set

$$z_1 = \gamma z + \delta,$$

(3) becomes

$$z' = \frac{\alpha}{\gamma} + \frac{\beta\gamma - \alpha\delta}{\gamma}\frac{1}{z_1},$$

and hence is the product of the three transformations

$$z' = \frac{\alpha}{\gamma} + \frac{\beta\gamma - \alpha\delta}{\gamma} z_2, \qquad z_2 = \frac{1}{z_1}, \qquad z_1 = \gamma z + \delta.$$

That these transformations are of types (1 a) and (2 a) is obvious.

It follows that all the transformations (3) are directly conformal and circular. They evidently form a group and, since the complex constants α, β, γ, δ enter homogeneously, the group depends on six real parameters.

THEOREM 1. *There exists a six-parameter group of directly conformal circular transformations, namely, the group of linear transformations, with complex coefficients, of $z = x + iy$ into $z' = x' + iy'$.*

A particular inversely conformal circular transformation is the reflection in the x-axis,

$$z' = \bar{z},$$

where $\bar{z} = x - iy$ is the conjugate of z. The product of it and the general transformation (3), namely

$$z' = \frac{\alpha\bar{z} + \beta}{\gamma\bar{z} + \delta}, \qquad \alpha\delta - \beta\gamma \neq 0, \tag{4}$$

is then also inversely conformal and circular.

THEOREM 2. *There exists a six-parameter set of inversely conformal circular transformations, namely, the set of linear transformations, with complex coefficients, of $\bar{z} = x - iy$ into $z' = x' + iy'$.*

The inversely conformal transformations (4) do not form a group, for the product of two of them is a directly conformal transformation (3). The transformations (3) and (4) taken together do form a group, and this is the group of *all* circular transformations.*

THEOREM 3. *The group of all circular transformations consists of*

* For a proof of the fact that there are no other circular transformations, see Coolidge, *A Treatise on the Circle and the Sphere*, p. 309.

$2 \cdot \infty^6$ *transformations*,* *the directly conformal transformations* (3) *and the inversely conformal transformations* (4).

For the sake of brevity we shall call the transformations (3) the *direct*, and the transformations (4) the *indirect*, circular transformations.

We have noted that a transformation (4) is the product of a reflection in a line and a transformation (3), and that a transformation (3) is of type (1 a) or (2 a), or a product of transformations of these types.

Henceforth, we shall revert to our previous convention of considering reflection in a line as an inversion. The facts just noted may then be summarized as follows.

THEOREM 4. *A circular transformation is an inversion, a transformation of similarity, or the product of transformations of these two kinds.*

We recall that a transformation of similarity is a radial transformation of similarity, a rotation, a translation, or the product of transformations of these three descriptions. But it is readily shown that each of these three transformations is the product of two inversions (Ex. 4). Hence:

THEOREM 5. *Every circular transformation is the product of a number of inversions, an even number if it is directly conformal, and an odd number if it is inversely conformal.*

The theorem brings out the importance of inversion and the simplicity of the structure of a circular transformation.

Conformal Collineations. The transformations (1) or

$$z' = \alpha z + \beta, \qquad \alpha \neq 0, \tag{5}$$

are precisely the collineations of the plane which preserve *directed* angle (Ch. XI, § 5, Ex. 3). Accordingly, we call them henceforth the *direct* transformations of similarity or the *directly conformal collineations*. On the other hand, the transformations

$$z' = \alpha \bar{z} + \beta, \qquad \alpha \neq 0, \tag{6}$$

are the collineations which preserve the magnitude of every angle

* The group (3) is a continuous group: it is possible to pass continuously from any given transformation $(\alpha_1, \beta_1, \gamma_1, \delta_1)$ of (3) through transformations of (3) to any prescribed transformation $(\alpha_2, \beta_2, \gamma_2, \delta_2)$ of (3). Similarily, the set (4) is a continuous set. But it is impossible to pass continuously from a directly conformal transformation (3) to an inversely conformal transformation (4) through transformations (3) and (4) alone. Hence, (3) and (4) present two *unconnected* continua of transformations, each dependent on six real parameters. It is this fact that we express in saying that the group which comprises both consists of $2 \cdot \infty^6$ transformations.

but reverse the sense. They are known as the *indirect* **transformations** of similarity or as the *inversely conformal collineations.*

The $2\cdot\infty^4$ transformations of similarity, (5) and (6), form a subgroup of the group of all circular transformations, (3) and (4), and the direct transformations (5) form a subgroup of the group (3).

Rigid Motions and Reflections. The rigid motions of the plane are the transformations (1) for which $r = 1$, that is, the transformations (5) for which $|\alpha| = 1$.* The transformations (6) for which $|\alpha| = 1$ are those obtained by forming the product of the reflection in the axis of x with each rigid motion. They are generally known as the *reflections of the plane.*

THEOREM 6. *The $2\cdot\infty^3$ rigid motions and reflections of the plane constitute a subgroup of the group of all circular transformations, and the rigid motions in themselves, a subgroup of the group of direct circular transformations.*

EXERCISES

1. Show that

$$x' = \frac{x^2 + y^2 - x}{(x-1)^2 + y^2}, \qquad y' = -\frac{y}{(x-1)^2 + y^2}$$

is a direct circular transformation. Determine the equivalent equation of the form (3).

2. Find the equations in the usual form of the transformation (3).

3. Show that every circular transformation carries two points mutually inverse in a circle into two points mutually inverse in the transformed circle.

4. Establish the following propositions.

(*a*) A rotation about a point O through the angle ϕ is the product of the reflections in two lines through O so chosen that the angle from the first to the second is $\phi/2$.

(*b*) A translation is the product of the reflections in two suitably chosen parallel lines.

(*c*) A homothetic transformation with finite center and positive ratio of similitude is the product of the inversions in two suitably chosen proper circles.

16. Inversive Geometry. A property which is invariant with respect to the group of direct circular transformations we shall call an *inversive property*, a theorem which involves only inversive properties an inversive theorem, and the geometry of inversive properties *inversive geometry.*

* By definition, if $\alpha = a_1 + i\,a_2$, $|\alpha| = \sqrt{a_1^2 + a_2^2}$. In the present case, $\alpha = r(\cos\phi + i\sin\phi)$, so that $|\alpha| = r$.

On the group of motions as a subgroup of the group of collineations, we based a subgeometry of projective geometry, which we called metric geometry. In precisely the same way, we now base on the group of motions, as a subgroup of the group of direct circular transformations, a subgeometry of inversive geometry which we shall call the *metric geometry of the circle.*

It is important to note that the term "metric property" is a *relative* term and takes on a meaning with respect to inversive geometry different from that which it had with respect to projective geometry. From the projective standpoint, a metric property is one invariant with respect to the metric group, but not with respect to all collineations.* From the inversive point of view, a metric property is one invariant with respect to the metric group, but not with respect to all direct circular transformations.

The two metric geometries differ also. The "projective-metric" geometry deals with properties defined as metric relative to the projective group, either alone or in conjunction with projective properties. The "inversive-metric" geometry has to do with properties defined as metric relative to the inverisve group, either by themselves, or in conjunction with inversive properties.

From the projective point of view, the circle is a metric figure and all properties and theorems pertaining to it are metric. From the inversive standpoint, the properties of a circle are in part inversive and in part metric. In other words, properties which we have previously looked upon as metric now fall into two groups, those which are metric from the new standpoint and those which are inversive. For example, a circle is now an inversive figure, whereas the fact that a point is the center of a circle is metric. Again, distance is still metric, but angle is inversive.

The content of Part A is half metric and half inversive. The essential results concerning orthogonal and tangent circles are inversive, whereas the concept of a power of a point is, for example, metric.

In projective geometry we frequently obtained projective theorems by generalizing metric theorems. In the same way it is possible, here, to obtain inversive theorems by generalizing metric theorems by inversion. Theorem 3 and Exercise 5 of § 13 are typical inversive generalizations of metric theorems of Part A.

* Or all affine transformations, if we adopt the finer projective classification of Ch. XI, § 6.

But this is not the most important metric road to inversive results. Inversive theorems may be obtained directly by metric methods, and frequently more easily in this than in any other way. The results of Part A are to a large extent inversive, but the idea of a power of a point which was so useful in developing them is metric.

Isotropic Coordinates. The equations

$$z = x + i\,y, \qquad \bar{z} = x - i\,y \tag{1}$$

represent a change from the Cartesian coordinates (x, y) to new coordinates $(z, \bar{z})$, known as *isotropic coordinates.*

Since for a real point z and $\bar{z}$ are conjugate-complex numbers, one of them suffices to determine the point. The standard choice for this purpose is z, as is evident from the preceding section.

The equations of a direct circular transformation in isotropic coordinates are

$$z' = \frac{\alpha\,z + \beta}{\gamma\,z + \delta}, \qquad \bar{z}' = \frac{\bar{\alpha}\,\bar{z} + \bar{\beta}}{\bar{\gamma}\,\bar{z} + \bar{\delta}}, \qquad \alpha\,\delta - \beta\,\gamma \neq 0. \tag{2}$$

Since we are in the real domain, the two equations are equivalent to one another and the first suffices to represent the transformation.

It is readily found that the equation of a circle in isotropic coordinates is of the form

$$a\,z\bar{z} + \beta\,z + \bar{\beta}\,\bar{z} + d = 0, \tag{3}$$

where a, d are real, and β, $\bar{\beta}$ conjugate-complex, numbers.

If, instead of the single isotropic coordinate z for a real finite point P, we introduce homogeneous coordinates (z_1, z_2) where $z_1/z_2 = z$, we obtain in the usual way homogeneous coordinates for the point at infinity: $(z_1, 0)$, $z_1 \neq 0$. More generally, the equations

$$\frac{z_1}{z_2} = z, \qquad \frac{\bar{z}_1}{\bar{z}_2} = \bar{z} \tag{4}$$

define a change from the coordinates $(z, \bar{z})$ of P to homogeneous isotropic coordinates $(z_1, z_2; \bar{z}_1, \bar{z}_2)$. If $z_2\bar{z}_2 \neq 0$, the point P is finite. If $z_2\bar{z}_2 = 0$, then $z_2 = 0$, $\bar{z}_2 = 0$ and we have coordinates $(z_1, 0; \bar{z}_1, 0)$ for the point at infinity.

In $(z_1, z_2; \bar{z}_1, \bar{z}_2)$ we have homogeneous coordinates or parameters of the *mixed* type discussed in Ch. XI, § 1. If $(z_1, z_2; \bar{z}_1, \bar{z}_2)$ is one set of coordinates for a point, all sets are given by $(\rho\,z_1, \rho\,z_2; \bar{\rho}\,\bar{z}_1, \bar{\rho}\,\bar{z}_2)$, $\rho\bar{\rho} \neq 0$.

Each of the equations (2) is replaced by two equations in the homo-

geneous isotropic coordinates. Equation (3) becomes

(5) $$a\, z_1\bar{z}_1 + \beta\, z_1\bar{z}_2 + \bar{\beta}\, z_2\bar{z}_1 + d\, z_2\bar{z}_2 = 0.$$

When $a = \beta = \bar{\beta} = 0$ and $d \neq 0$, this equation reduces to $z_2\bar{z}_2 = 0$ and represents the point at infinity, considered as a null circle.

EXERCISES

1. Classify the following figures as inversive or metric.

(*a*) Two points mutually inverse in a circle.
(*b*) Two proper circles and their radical axis.
(*c*) An orthogonal system of circles.
(*d*) Three proper circles and their radical center.
(*e*) Two proper circles and their centers of similitude.
(*f*) Two circles and one of their circles of antisimilitude.

2. The same for the following theorems. *

(*a*) § 5, Ex. 4; § 6, Th. 1 *a*, 1 *b*, 7, 8.
(*b*) § 7, Th. 3; § 9, Th. 4; § 10, Ths. 2, 4, 6.
(*c*) § 12, Ex. 5; § 13, Th. 1; § 14, Ths. 2, 3.

3. Establish equation (3).

17. Cross Ratio in Inversive Geometry. The cross ratio (P_1P_2, P_3P_4) of four distinct finite points $P_k : z_k = x_k + i\, y_k$, $k = 1, 2, 3, 4$, is defined as

(1) $$(P_1P_2,\, P_3P_4) = \frac{(z_3 - z_1)(z_4 - z_2)}{(z_3 - z_2)(z_4 - z_1)}\,.$$

Four distinct points have 24 cross ratios, connected by the usual relations. Since a direct circular transformation is a linear transformation of z into z', each cross ratio is an inversive invariant.

THEOREM 1. *The cross ratios of four points are invariant with respect to the group of direct circular transformations.*

The cross ratio

$$\lambda = (P_1P_2,\, P_3P_4), \qquad \lambda \neq 0, 1,$$

is a complex number. We write it in the so-called polar form

(2) $$\lambda = r\, e^{\theta i} \qquad \text{or} \qquad \lambda = r\,(\cos\theta + i \sin\theta),$$

in which the *arc* or *argument* θ admits an additive integral multiple of $2\,\pi$, whereas the *absolute value* r is fixed.

The absolute value of a product (quotient) of two complex numbers is the product (quotient) of their absolute values, and the arc of a

* If a theorem falls into two parts, one of which is metric and the other inversive, the fact should be noted.

product (quotient) of two complex numbers is the sum (difference) of their arcs, provided the determinations of the three arcs are suitably chosen. Thus

$$r = \frac{|z_1 - z_3| \cdot |z_2 - z_4|}{|z_2 - z_3| \cdot |z_1 - z_4|}, \qquad \theta = \text{arc}\,\frac{z_1 - z_3}{z_2 - z_3} + \text{arc}\,\frac{z^2 - z_4}{z_1 - z_4}\cdot$$

Since, for example, $|z_1 - z_3|$ is the absolute distance P_3P_1 and arc $(z_1 - z_3)$ is the directed angle from the positive direction of the x-axis to that of the directed line-segment $\overline{P_3P_1}$, we have:

THEOREM 2. *The absolute value and arc of the cross ratio* $(P_1P_2,\ P_3P_4)$ *have respectively the values*

$$(3) \quad r = \frac{P_3P_1 \cdot P_4P_2}{P_3P_2 \cdot P_4P_1}, \qquad \theta \equiv \angle\, P_2P_3P_1 + \angle\, P_1P_4P_2 \ (\text{mod. } 2\,\pi),^*$$

where $\angle\, P_2P_3P_1$, *for example, is the directed angle from* $\overline{P_3P_2}$ *to* $\overline{P_3P_1}$.

It is at once evident that, if the four points P_k lie on a circle, $\theta \equiv 0$ (mod. π), and conversely. But $\theta \equiv 0$ (mod. π) when and only when λ is real and, if one of the cross ratios of four points is real, all are.

THEOREM 3. *A necessary and sufficient condition that four points be concyclic is that an arbitrary one of their cross ratios be real.*

THEOREM 4. *If* P_3, P_4 *lie on a circle* C_{34} *in which* P_1, P_2 *are mutually inverse, then* P_1, P_2 *lie on a circle* C_{12} *in which* P_3, P_4 *are mutually inverse.*

The two pairs of points are said to be *orthocyclic*: C_{34} is orthogonal to every circle through P_1, P_2 and C_{12} is orthogonal to every circle through P_3, P_4. The theorem is a consequence of § 5, Ex. 2. By hypothesis, $P_3P_1/P_3P_2 = P_4P_1/P_4P_2$; hence $P_1P_3/P_1P_4 = P_2P_3/P_2P_4$ and the conclusion follows. Incidentally, we see that the condition for orthocyclic position is that $r = 1$.

THEOREM 5. *The pairs of points* P_1, P_2 *and* P_3, P_4 *are orthocyclic if and only if the absolute value of* $(P_1P_2,\ P_3P_4)$ *is unity.*

As in projective geometry, we say that P_1, P_2 and P_3, P_4 are *harmonic* when $(P_1P_2,\ P_3P_4) = -1$. Hence:

THEOREM 6. *A necessary and sufficient condition that two pairs of points be harmonic is that they be both concyclic and orthocyclic.*

* We mean by this notation that θ is equal to the sum of the two angles in question to within an additive integral multiple of $2\,\pi$. In general, $a \equiv b$ (mod. 2π): *a is congruent to b, modulus* $2\,\pi$, if $a = b + 2\,k\pi$ where k is an integer.

THEOREM 7. *Two points are harmonic with two given points if and only if they are the points of intersection of a circle on which the given points lie with a circle in which the given points are mutually inverse.*

We have restricted the definition of cross ratio and the proofs of the theorems to the case of four finite points. Both definition and proofs can, however, be readily extended to the case in which one of the four points is the point at infinity.*

Direct Circular Transformations. A point transformation of the plane of inversion which is one-to-one and preserves cross ratio must, by Th. 3, carry concyclic points into concyclic points and is, therefore, a circular transformation. In fact, if it carries the points P_1, P_2, P_3 into the points P_1', P_2', P_3' and the arbitrary point P into P', it is represented geometrically by the equation

$$(P_1'P_2', P_3'P') = (P_1P_2, P_3P), \tag{4}$$

and hence analytically by an equation of the form

$$z' = \frac{\alpha z + \beta}{\gamma z + \delta}, \qquad \alpha\delta - \beta\gamma \neq 0. \tag{5}$$

It is thus not only circular, but directly conformal.

THEOREM 8. *A necessary and sufficient condition that a one-to-one point transformation of the real plane of inversion be a direct circular transformation is that it preserve cross ratio.*

If all the points of (4), except P', are given, then P' is uniquely determined. Hence:

THEOREM 9. *There is a unique direct circular transformation which carries three given distinct points into three prescribed distinct points.*

A direct circular transformation, other than the identity, has two real fixed points, as may be shown by applying the usual method to (5). If the two fixed points coincide, the transformation is said to be *parabolic*.

We restrict ourselves to the case of a *nonparabolic* transformation as the one of principal interest. If P_1, P_2 are the fixed points and $P \rightarrow P'$

* The point at infinity is included if we employ homogeneous coordinates (z_1, z_2) instead of z. However, the extension of the proofs is simpler if we abide by the coordinates in the text and employ a limiting process. If P_4, for example, recedes without limit, $(z_4 - z_2)/(z_4 - z_1)$ approaches unity. Then

$$\lambda = \frac{z_3 - z_1}{z_3 - z_2}, \qquad r = \frac{P_3P_1}{P_3P_2}, \qquad \theta \equiv \angle\, P_2P_3P_1 \pmod{2\pi}.$$

an arbitrary pair of corresponding points, the cross ratio $(P_1P_2, P\,P')$ is readily proved to be constant:

$$(P_1P_2, P\,P') = \lambda, \qquad \lambda \neq 0, 1. \tag{6}$$

The constant λ is known as the *invariant* of the transformation.

We recognize three different types of nonparabolic transformations.

The *hyperbolic* transformations: λ real. By Th. 3, corresponding points are concyclic with the fixed points. Hence, each circle of the pencil S_1 of circles through the fixed points is carried into itself.

The *elliptic* transformations: $|\lambda| = 1$. Corresponding points are orthocyclic with the fixed points. Each circle of the pencil S_2 of circles in which the fixed points are mutually inverse remains fixed.

The *loxodromic* transformations: λ not real and $|\lambda| \neq 1$. It may be shown (Ex. 11) that corresponding points lie on a double spiral which curls about each of the fixed points and cuts every circle of the pencil S_1 (or S_2) at a fixed angle. Each of these ∞^1 double spirals is carried into itself.

A transformation for which $\lambda = -1$ is involutory. Two corresponding points are harmonic with the fixed points. As a matter of fact, the transformation is both hyperbolic and elliptic, every circle of the orthogonal system formed by the pencils S_1 and S_2 is carried into itself, and the mate of a point P is the second intersection P' of the two circles of the system which pass through P.

It is readily shown that the transformations of this type are the only direct circular transformations which are involutory. They are known as *Moebius involutions*.

General Isotropic Coordinates. If P_*, P_0, P_1 are three arbitrarily chosen fixed points and P an arbitrary point, the generalized isotropic coordinates of P consist of two complex numbers $(z, \bar{z})$ where z is defined as the cross ratio

$$z = (P_*P_0, P_1P)$$

and $\bar{z}$ is the conjugate of z.

It may be proved, by means of the methods of Ch. IX, § 2, that the original isotropic coordinates are a special case of the general isotropic coordinates, that the equations of a change from one system of isotropic coordinates to a second are of the type (2) of § 16, and that real inversive geometry has the same analytic form in terms of the general isotropic coordinates as in terms of the original ones.

In the general system of isotropic coordinates the point at infinity

loses its identity as the point without coordinates. Moreover, the point which is now lacking coordinates, the point P_*, may be chosen arbitrarily. Thus, all points of the plane are on the same footing; no point plays a special rôle.

THEOREM 10. *The real inversive plane is a closed continuum without an exceptional point.*

It follows that in inversive geometry the straight lines, that is, the circles through the point at infinity, are just as much "proper" circles as the proper circles themselves.

The Projective Equivalent of Real Inversive Geometry. It is evident from the foregoing developments that the real inversive geometry of the plane is analytically identical with the complex projective geometry of a line. There exists a perfect one-to-one correspondence between the two geometries. To the complex points $z = x + i\,y$ of the line correspond the real points $z = x + i\,y$ of the plane, and to the complex projective transformations of the line correspond the real direct circular transformations of the plane.

The correspondence enables us to represent each geometry in terms of the other. It is especially useful in the study of the complex projective geometry of the line in that it furnishes for this geometry a *real representation*, that is, an equivalent *real* geometry.*

EXERCISES

1. Show that, if an indirect circular transformation carries P_1, P_2, P_3, P_4 into P_1', P_2', P_3', P_4', the cross ratio $\lambda' = (P_1'P_2', P_3'P_4')$ is conjugate-complex to the cross ratio $\lambda = (P_1P_2, P_3P_4) : \lambda' = \bar{\lambda}$; hence prove that $|\lambda'| = |\lambda|$ and arc $\lambda' \equiv -$ arc λ (mod. $2\,\pi$).

2. Prove geometrically that inversion preserves the absolute value of cross ratio and preserves the arc of cross ratio except for sign.

3. Enlarge on the proof of Theorem 3. Show also that two concyclic pairs of points P_1, P_2 and P_3, P_4 separate or do not separate one another according as (P_1P_2, P_3P_4) is negative or positive.

4. What is the locus of a point which moves so that it and a given point are always orthocyclic to two given points?

5. Prove Theorem 7.

6. Find the fixed points and invariants of the transformations:

$$(a)\quad z' = \frac{i\,z + 2(1 + i)}{(i - 1)z + 3}, \qquad (b)\quad z' = 4\,z.$$

* For a detailed discussion of the correspondence, see Coolidge, *The Geometry of the Complex Domain*, Ch. II.

7. Show that the transformation of § 15, Ex. 1 is a Moebius involution. What are the fixed points?

8. Prove that a reflection in a finite point is a Moebius involution. Show that every Moebius involution is the product of the inversions in any two circles which pass through the fixed points of the involution and are mutually orthogonal.

9. Show that, by a proper choice of isotropic coordinates, the equation of a nonparabolic direct circular transformation may be reduced to the canonical form

$$z' = \lambda z, \qquad \lambda \neq 0, 1,$$

and that of a parabolic transformation to the canonical form

$$z' = z + \beta, \qquad \beta \neq 0.$$

10. Show that a parabolic transformation carries into itself each of the circles of a pencil of circles mutually tangent at the fixed point of the transformation.

11. Prove the facts stated in the text concerning loxodromic transformations. Consider first the general loxodromic transformation with the origin and point at infinity as fixed points, and show that in this case two corresponding points lie on an ordinary equiangular spiral.

12. Prove that the directly conformal collineations are precisely the direct circular transformations which leave fixed the point at infinity. Hence justify the statement: *The real metric geometry of the circle is essentially the subgeometry of real inversive geometry in which a certain real point is held fast.*

18. The Complex Plane of Inversion. We recall the construction of the *finite* complex plane (Ch. VIII, § 2). Two ordered complex numbers (x, y) are given. When real, they are the rectangular coordinates of a real point. When not both are real, they are assigned as coordinates to a newly created, imaginary, point. The totality of points (x, y), real and imaginary, constitute the finite complex plane.

For the study of inversive geometry in the *finite* complex plane we employ, instead of (x, y), the *isotropic coordinates* (u, v):

$$\begin{aligned} u &= x + i\,y, & v &= x - i\,y, \\ x &= \frac{1}{2}(u + v), & y &= \frac{1}{2\,i}(u - v). \end{aligned} \tag{1}$$

Evidently, the point (u, v) is real when and only when u and v are conjugate-complex. In fact, for real points the coordinates (u, v) are precisely the coordinates $(z, \bar{z})$ of § 16.

It is clear from (1) that the equations

$$u = k, \qquad v = l, \tag{2}$$

where k and l are arbitrary complex constants, represent respectively

the two families of isotropic lines. It is from this fact that the coordinates (u, v), and also $(z, \bar{z})$, take their name.

The Complex Plane of Inversion. We replace (u, v) by the homogeneous coordinates $(u_1, u_2; v_1, v_2)$ defined by the relations

$$(3) \qquad \frac{u_1}{u_2} = u, \qquad \frac{v_1}{v_2} = v.$$

Equations (2) become

$$(4\,a) \quad k_1u_1 + k_2u_2 = 0, \quad k_1 \neq 0, \qquad l_1v_1 + l_2v_2 = 0, \quad l_1 \neq 0.$$

Two quadruples $(u_1, u_2; v_1, v_2)$ whose corresponding number pairs are proportional we shall call *related* quadruples, provided no pair of either quadruple is the pair 0, 0. The quadruples, one or both of whose pairs is 0, 0, we discard once and for all.

A quadruple $(u_1, u_2; v_1, v_2)$ for which $u_2v_2 \neq 0$ and the related quadruples $(\rho u_1, \rho u_2; \sigma v_1, \sigma v_2)$, $\rho\sigma \neq 0$, are the sets of homogeneous coordinates of the finite point $u = u_1/u_2$, $v = v_1/v_2$; and the quadruples $(\rho, 0; \sigma, 0)$, $\rho\sigma \neq 0$, those of the real point at infinity.

The remaining sets of related quadruples are those for which $u_2 = 0$, $v_2 \neq 0$ or $v_2 = 0$, $u_2 \neq 0$. Corresponding to these sets we create new points, which for obvious reasons we call *the imaginary points at infinity* and to which we assign the sets as coordinates.

The points at infinity are evidently the points whose coordinates satisfy one or the other—or, in case of the real point at infinity, both—of the equations

$$(4\,b) \qquad u_2 = 0, \qquad v_2 = 0.$$

These equations are identical with the equations (4 a) for which $k_1 = 0$ $(k_2 \neq 0)$ and $l_1 = 0$ $(l_2 \neq 0)$. Accordingly, we call their loci the isotropic lines at infinity or *the infinite isotropic lines* and adjoin them respectively to the two families of finite isotropics (4 a).

The equations (4 a) are satisfied respectively by the coordinates $(k_2, -k_1; 1, 0)$, $k_1 \neq 0$ and $(1, 0; l_2, -l_1)$, $l_1 \neq 0$. But these coordinates represent precisely the imaginary points on the lines $v_2 = 0$ and $u_2 = 0$ respectively. Accordingly, we think of the imaginary points on an infinite isotropic of one family of isotropics as lying one each on the finite isotropics of the other family.

Summary. *Corresponding to each finite isotropic there is created an imaginary point at infinity. Each imaginary point at infinity lies on the corresponding finite isotropic and on no other finite line. The imaginary*

points at infinity which lie on the finite isotropics of one family of isotropics form with the real point at infinity the infinite isotropic of the other family. Thus, the infinite domain consists of two conjugate-imaginary isotropics whose common point is the real point at infinity.

The totality of complex points, finite and at infinity, constitutes the *complex plane of inversion.*

An important property of this plane is obvious from our construction of it.

THEOREM 1. *Through each point of the complex inversive plane there pass two isotropic lines, one of each kind.*

The Real Geometry of the Complex Inversive Plane. The equation of the real circle

$$(5) \qquad a_1(x^2 + y^2) + a_2x + a_3y + a_4 = 0$$

becomes, in homogeneous isotropic coordinates,

$$(6) \qquad a\, u_1v_1 + \beta\, u_1v_2 + \bar\beta\, u_2v_1 + d\, u_2v_2 = 0,$$

where

$$a = a_1, \qquad \beta = \tfrac{1}{2}(a_2 - i\, a_3), \qquad \bar\beta = \tfrac{1}{2}(a_2 + i\, a_3), \qquad d = a_4.$$

Hence

$$(7) \qquad a_2^2 + a_3^2 - 4\, a_1a_4 \equiv 4\,(\beta\bar\beta - a\, d).$$

The classification of circles in the inversive plane may be obtained, by use of (7), from the classification of circles in the Cartesian plane (§ 1), provided due account is taken of the change in the infinite domain.

We note, first, that this change produces a radical effect on the concept of a degenerate circle. In the Cartesian plane, the circle (5) is degenerate if the left-hand side of (5), written in homogeneous Cartesian coordinates, is factorable into linear factors, that is, if $a_1(a, a) = 0$; it is, then, a null circle, a finite line circle, or the infinite line circle. In the inversive plane, the circle (6) is degenerate if the left-hand side of (6) is factorable into two factors linear respectively in u_1, u_2 and v_1, v_2; it consists, then, of two isotropic lines, and is always a null circle.

Of the degenerate circles of the Cartesian plane, the null circles and the infinite line circle: $(a, a) = 0$, become null, that is, degenerate, circles in the inversive plane. In particular, the infinite line circle is replaced by the new infinite domain: $u_2v_2 = 0$.

On the other hand, the finite line circles of the Cartesian plane: $a_1 = 0$, $(a, a) \neq 0$, are replaced by the real straight lines: $a = 0$, $\beta\bar{\beta} - a\,d \neq 0$, of the plane of inversion. These are nondegenerate, or proper, circles and, since $\beta\bar{\beta} - a\,d = \beta\bar{\beta} > 0$, they belong with the circles with real traces.

It is evident from the discussion that $\beta\bar{\beta} - a\,d = 0$ is the condition that the circle (6) be degenerate, and that a proper circle (6) has, or has not, a real trace according as $\beta\bar{\beta} - a\,d > 0$ or < 0.

THEOREM 2. *In the complex plane of inversion there are real circles of three types: the circles with real traces including the real straight lines, the null circles, and the circles without real traces. Equation* (6) *represents a circle of the first, second, or third type according as* $\beta\bar{\beta} - a\,d > 0$, $= 0$, *or* < 0.

THEOREM 3. *The infinite domain of the complex inversive plane is a null circle.*

We turn to the real circular transformations, agreeing to apply them to imaginary as well as to real points. The equations of the group of direct transformations in our present coordinates are

$$(8) \qquad \begin{aligned} \rho u_1' &= \alpha\, u_1 + \beta\, u_2, & \sigma v_1' &= \bar{\alpha}\, v_1 + \bar{\beta}\, v_2,\\ \rho u_2' &= \gamma\, u_1 + \delta\, u_2, & \sigma v_2' &= \bar{\gamma}\, v_1 + \bar{\delta}\, v_2, \end{aligned}$$

and those of the set of indirect transformations,

$$(9) \qquad \begin{aligned} \rho u_1' &= \alpha\, v_1 + \beta\, v_2, & \sigma v_1' &= \bar{\alpha}\, u_1 + \bar{\beta}\, u_2,\\ \rho u_2' &= \gamma\, v_1 + \delta\, v_2, & \sigma v_2' &= \bar{\gamma}\, u_1 + \bar{\delta}\, u_2, \end{aligned}$$

where in each case $\alpha\,\delta - \beta\,\gamma \neq 0$ and $\rho\,\sigma \neq 0$.

The very forms of the transformations reveal important facts.

THEOREM 4. *A real circular transformation is one-to-one without exception in the complex plane of inversion.*

THEOREM 5. *A real circular transformation carries an isotropic into an isotropic, in particular, into an isotropic of the same or opposite family according as it is directly or inversely conformal.*

Equations (8) may be interpreted as a change from the special isotropic coordinates $(u_1, u_2;\ v_1, v_2)$ to general isotropic coordinates $(u_1', u_2';\ v_1', v_2')$. Thereby the infinite domain loses its identity and assumes the same aspect as any other null circle.

A realization of inversive geometry on the sphere and a new type of coordinates for the inversive plane will be found in Ch. XIX, §§ 12, 13.

EXERCISES

1. Show that two isotropics of the same family never intersect, and that two isotropics of different families always intersect in a single point.

2. Prove that two real circles always intersect in two points, real and distinct, real and coincident, or conjugate-imaginary.

3. Prove directly that the bilinear form in equation (6) is degenerate if and only if $\beta \bar{\beta} - a d = 0$.

4. Show that, if the circle (6) is a null circle, coordinates of its center are $(\bar{\beta}, -a; \beta, -a)$ or $(d, -\beta; d, -\bar{\beta})$.

5. The usual rectangular coordinates (x, y) of the real finite plane are replaced by homogeneous coordinates $(x_1, x_2; y_1, y_2)$ defined by the relations

$$\frac{x_1}{x_2} = x, \qquad \frac{y_1}{y_2} = y.$$

Construct the real extended plane based on the coordinates $(x_1, x_2; y_1, y_2)$ and discuss in this plane the locus of the general equation with real coefficients which is bilinear in x_1, x_2 and y_1, y_2.

19. Inversion in the Complex Cartesian Plane. Inversion is one-to-one without exception in the complex inversive plane (§ 18, Th. 4). In the real Cartesian plane it fails of being one-to-one in that to the center of inversion corresponds every point at infinity (§ 8, Th. 1). In the complex Cartesian plane, in which we shall now consider it, it possesses a still different aspect.

A quick and instructive way of arriving at the facts in this case is to start with the geometric definition of inversion suggested by § 3, Ex. 1: *Two points are mutually inverse in a proper circle if and only if they are conjugate points with respect to the circle and are collinear with its center.*

Let O, I, J be respectively the center of inversion and the circular points at infinity (Fig. 13). Since O and each point conjugate to O, that is, each point on IJ, are collinear with O, every point on the line at infinity is inverse to O. Again, since the points conjugate to I are the points of the isotropic OI, each point of OI is inverse to I. Similarly, each point on the isotropic OJ is inverse to J.

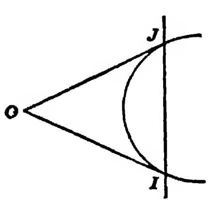


FIG. 13

The point, or points, P' inverse to a given point P are the points common to the polar of P and the line OP. There is, then, just one point P' inverse to P unless the line OP is undefined or coincides with the polar of P. In the first case P must be O, and in the second case P is either I or J.

Thus a point P, other than O, I, J, has a unique inverse. In particular, the inverse of a point at infinity, not I or J, is O, and the inverse of a point on OI (OJ), not O or I (O or J), is I (J).

The Projective Generalization of Inversion. This transformation is based on a nondegenerate conic C and a point A_3 not on C. Two points correspond by it when they are conjugate with respect to C and collinear with A_3.

Here, the rôles of the point O and the points I, J are played by the point A_3 and the points of contact A_1, A_2 of the tangents to C from A_3 (Fig. 14).

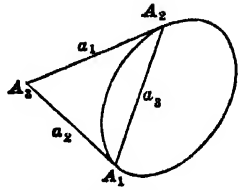


FIG. 14

THEOREM 1. *The points of the complex plane, other than those which lie on the sides of the triangle $A_1A_2A_3$, correspond in pairs. To a vertex of the triangle correspond all the points on a side: to A_3 all the points on a_3, and to $A_1(A_2)$ all the points on $a_2(a_1)$; and to a point on a side, other than a vertex, corresponds a vertex: to a point on a_3, not A_1 or A_2, the vertex A_3, and to a point on $a_1(a_2)$, not A_3 or $A_2(A_3$ or $A_1)$, the vertex $A_2(A_1)$.*

When we introduce projective coordinates referred to the triangle $A_1A_2A_3$ as triangle of reference and a point on C as unit point, the equation of C is

(1) $$x_1x_2 - x_3^2 = 0.$$

The two points $P : (x_1, x_2, x_3)$ and $P' : (x_1, x_2', x_3')$ are conjugate with respect to C and collinear with A_3 when

$$x_2x_1' + x_1x_2' - 2\,x_3x_3' = 0, \qquad x_2x_1' - x_1x_2' = 0.$$

Thus, the equations of the correspondence are found to be

$$\begin{aligned} &(2\,a) \qquad \rho x_1' = x_1x_3, \qquad \rho x_2' = x_2x_3, \qquad \rho x_3' = x_1x_2, \\ &(2\,b) \qquad \rho x_1 = x_1'x_3', \qquad \rho x_2 = x_2'x_3', \qquad \rho x_3 = x_1'x_2', \end{aligned} \qquad \rho \neq 0.$$

If P is a vertex of the triangle, for example, the vertex $A_3 : (0, 0, 1)$, equations (2 a) reduce to $x_1' = x_2' = x_3' = 0$, as might have been expected from the nature of the transformation. However, equations (2 b) become $x_1'x_3' = 0$, $x_2'x_3' = 0$, $x_1'x_2' = \rho$ and yield the proper information.

The transformation carries

$$a_1x_1 + a_2x_2 + a_3x_3 = 0$$

into

$$a_1x_1'x_3' + a_2x_2'x_3' + a_3x_1'x_2' = 0.$$

THEOREM 2. *The transformation carries a straight line into a conic through the vertices of the triangle. The conic is nondegenerate or degenerate according as the line does not, or does, contain a vertex of the triangle. In particular, if the line is a side of the triangle, the conic consists of two sides of the triangle.*

The fact that the conic goes through the vertices of the triangle may also be seen geometrically. Consider, for example, the general case in which the given line does not contain a vertex. The line then intersects the sides in three distinct points, and these points are carried into the three vertices.

A nondegenerate conic intersects each side of the triangle in two points, distinct or coincident. Hence the curve of the fourth order into which it is carried has a double point or cusp in each of the vertices of the triangle. According as the conic contains no, one, two, or all three of the vertices of the triangle, no, one, two, or all three sides of the triangle are constituent parts of the curve of the fourth order, and conversely. It follows that this curve, in the four cases, is a nondegenerate biquadratic, a nondegenerate cubic and one side of the triangle, a nondegenerate conic and two sides, and a straight line and all three sides. Consider, for example, the first case, and suppose that the biquadratic degenerated, say, into a nondegenerate cubic and a straight line. Since the biquadratic has singular points in all three vertices, the cubic would have to have its maximum of one singular point in a vertex of the triangle and the straight line would have to be the opposite side. Hence the given conic would contain a vertex of the triangle,—a contradiction. Again, if the biquadratic degenerated into two nondegenerate conics, these conics would pass through the vertices of the triangle and the given conic would be degenerate.

THEOREM 3. *The transformation carries a nondegenerate conic into a curve of the fourth order with a double point or cusp at each vertex of the triangle. This curve is nondegenerate if the given conic does not contain a vertex of the triangle, and degenerates into a nondegenerate cubic, a nondegenerate conic, or a straight line, taken respectively with one, two, or all three sides of the triangle, according as the given conic contains one, two, or all three vertices of the triangle.*

Since inversion is a metric case of our transformation (2 a), Theorems 2 and 3 may be readily interpreted for it.

The transformation (2 a) is, *in general*, one-to-one, and x_1', x_2', x_3'

are proportional to three quadratic polynomials in x_1, x_2, x_3 which have no linear factor in common. Any point transformation of the projective or Cartesian plane which has these properties is known as a *quadratic transformation.*

EXERCISES

1. Making use of Theorems 2, 3, discuss the effect of inversion on straight lines and proper circles. Consider all cases.

2. Prove geometrically and analytically, in the case of the transformation (2 a), that when the point P approaches A_3 along a given line through A_3, the corresponding point P' approaches a definite point on a_3.

3. When does the curve of the fourth order of Theorem 3 have a cusp at a given vertex of the triangle?

4. Generalize Pascal's Theorem by means of the transformation (2 a).

5. Discuss the quadratic transformation

$$\rho x_1' = x_2x_3, \qquad \rho x_2' = x_3x_1, \qquad \rho x_3' = x_1x_2, \qquad \rho \neq 0.$$

6. Show that the transformation of a point into its isogonal conjugate with respect to a triangle (Ch. III, § 10, Ex. 5) is a quadratic transformation and find its equations in terms of trilinear coordinates (Ch. X, § 7).

7. Prove that the transformation of a point into its conjugate with respect to the conics of a pencil of point conics of Type I is a quadratic transformation. What light does this throw on the content of Ch. XVI, § 16?

8. Generalize the transformation of inversion by polar reciprocation.

CHAPTER XIX

SPACE GEOMETRY

We shall be concerned primarily in this chapter with the development of projective geometry and its subgeometries in three-dimensional space. The subject matter will be analogous to that in the plane. The methods of treatment, however, we shall seek to vary as much as possible.

1. Extended Space. Homogeneous Cartesian Coordinates. The extension of the finite domain to a closed continuum has already been discussed in Ch. II, § 3. If (x, y, z) are rectangular Cartesian coordinates of a finite point P, homogeneous coordinates (x_1, x_2, x_3, x_4) of P are defined by the relations

$$(1) \qquad \frac{x_1}{x_4} = x, \qquad \frac{x_2}{x_4} = y, \qquad \frac{x_3}{x_4} = z.$$

If L is any line in the direction with components l, m, n and (x_0, y_0, z_0) is a fixed point on L, an arbitrary finite point P on L has the nonhomogeneous coordinates $(x_0 + l\,t,\ y_0 + m\,t,\ z_0 + n\,t)$, and hence the homogeneous coordinates $(x_0/t + l,\ y_0/t + m,\ z_0/t + n,\ 1/t)$. When P recedes indefinitely on L, the parameter t becomes infinite, and the latter coordinates approach $(l, m, n, 0)$ as limits.

Agreement. *The point at infinity in the direction with components* x_1, x_2, x_3 *shall have the homogeneous coordinates* $(x_1, x_2, x_3, 0)$.

It follows that the equation of the plane at infinity is

$$(2) \qquad x_4 = 0.$$

It is now true that *every* linear homogeneous equation in x_1, x_2, x_3, x_4, not all of whose coefficients are zero, represents a plane, and conversely.

Homogeneous Cartesian coordinates of a plane shall be defined as the ordered coefficients in an equation of the plane. Thus, homogeneous coordinates of the plane

$$a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4 = 0$$

are (a_1, a_2, a_3, a_4) or $(\rho a_1, \rho a_2, \rho a_3, \rho a_4)$, $\rho \neq 0$.

THEOREM 1. *The point $x : (x_1, x_2, x_3, x_4)$ lies in the plane $u : (u_1, u_2, u_3, u_4)$ if and only if*

$$(u|x) = 0, \tag{3}$$

where

$$(u|x) \equiv u_1x_1 + u_2x_2 + u_3x_3 + u_4x_4.$$

The equation

$$(a|u) \equiv a_1u_1 + a_2u_2 + a_3u_3 + a_4u_4 = 0,$$

where the a's are constants, not all zero, is satisfied by the coordinates (u_1, u_2, u_3, u_4) of those and only those planes which pass through the point $a : (a_1, a_2, a_3, a_4)$. It is, then, the equation in plane coordinates of the point a.

THEOREM 2. *Every linear homogeneous equation in plane coordinates, not all of whose coefficients are zero, represents a point, and conversely.*

2. Points, Planes, and Lines. Duality. Let there be given a number of distinct points and the same number of distinct planes, for example:

POINTS		PLANES	
Coordinates	*Equations*	*Coordinates*	*Equations*
$a : (a_1, a_2, a_3, a_4)$	$(a\|u) = 0$	$a : (a_1, a_2, a_3, a_4)$	$(a\|x) = 0$
$b : (b_1, b_2, b_3, b_4)$	$(b\|u) = 0$	$b : (b_1, b_2, b_3, b_4)$	$(b\|x) = 0.$

The following theorems are readily proved.

THEOREM 1 *a*. *Two distinct points determine a line. Equations in plane coordinates of the line determined by the points a, b are*

$$(a|u) = 0, \quad (b|u) = 0. \tag{1 a}$$

THEOREM 1 *b*. *Two distinct planes determine a line. Equations in point coordinates of the line determined by the planes a, b are*

$$(a|x) = 0, \quad (b|x) = 0. \tag{1 b}$$

In equations (1 b), we have a system of two linear homogeneous equations in four unknowns, which is of rank two. The simultaneous solutions of the equations are, then, the linear combinations of two nonproportional particular solutions (c_1, c_2, c_3, c_4), (d_1, d_2, d_3, d_4). Hence the points of a line are the linear combinations of two distinct points on the line.

The same considerations applied to equations (1 a) tell us that the planes through a line are the linear combinations of two distinct planes through the line. This totality of planes is known as a *pencil*

of planes. The line is called the *axis* of the pencil and three or more planes of the pencil are said to be *coaxal.*

THEOREM 2 a. *The equation*

$$k(a|u) + l(b|u) = 0,$$

where k, l are parameters not vanishing simultaneously, represents the range of points determined by the points a, b. The coordinates of an arbitrary point of the range are

$$k\,a + l\,b.$$

THEOREM 2 b. *The equation*

$$k(a|x) + l(b|x) = 0,$$

where k, l are parameters not vanishing simultaneously, represents the pencil of planes determined by the planes a, b. The coordinates of an arbitrary plane of the pencil are

$$k\,a + l\,b.$$

A finite number of points (planes) are said to be linearly dependent if their sets of *homogeneous* coordinates are linearly dependent. By Theorem 2:

THEOREM 3 a. *Three points are linearly dependent if and only if they are collinear.*

THEOREM 3 b. *Three planes are linearly dependent if and only if they are coaxal.*

If the three points a, b, c are not collinear, the system of equations

$$(a|u) = 0, \qquad (b|u) = 0, \qquad (c|u) = 0$$

is of rank three and every solution of it is a multiple of the particular solution

(2) $$|a_2 b_3 c_4|, \quad -|a_1 b_3 c_4|, \quad |a_1 b_2 c_4|, \quad -|a_1 b_2 c_3|.$$

Hence, there is a unique plane containing the three points, and its equation is

$$|a_2 b_3 c_4|x_1 - |a_1 b_3 c_4|x_2 + |a_1 b_2 c_4|x_3 - |a_1 b_2 c_3|x_4 = 0,$$

or

$$|a_1 b_2 c_3 x_4| = 0.$$

THEOREM 4 a. *Three non-collinear points a, b, c determine a plane. The plane has the coordinates* (2) *and the equation*

(3 a) $$|a\,b\,c\,x| = 0.$$

THEOREM 4 b. *Three non-coaxal planes a, b, c determine a point. The point has the coordinates* (2) *and the equation*

(3 b) $$|a\,b\,c\,u| = 0.$$

The following theorems may now be readily established.

THEOREM 5 a. *The points*

$$k\,a + l\,b + m\,c$$

THEOREM 5 b. *The planes*

$$k\,a + l\,b + m\,c$$

linearly dependent on the three noncollinear points a, b, c are the points of the plane determined by the three points.

linearly dependent on the three noncoaxal planes a, b, c are the planes through the point determined by the three planes.

The totality of planes through a point is known as a *sheaf of planes.*

THEOREM 6 *a*. *The four points a, b, c, d are coplanar, that is, lie in a plane, if and only if*

$$|a\,b\,c\,d| = 0.$$

THEOREM 6 *b*. *The four planes a, b, c, d are copunctual, that is, go through a point, if and only if*

$$|a\,b\,c\,d| = 0.$$

THEOREM 7 *a*. *Four points are linearly dependent if and only if they are coplanar.*

THEOREM 7 *b*. *Four planes are linearly dependent if and only if they are copunctual.*

Five sets of four numbers each are always linearly dependent. Hence:

THEOREM 8 *a*. *Five points are always linearly dependent.*

THEOREM 8 *b*. *Five planes are always linearly dependent.*

THEOREM 9 *a*. *The points*

$$k\,a + l\,b + m\,c + n\,d$$

linearly dependent on four noncoplanar points a, b, c, d are all the points of space.

THEOREM 9 *b*. *The planes*

$$k\,a + l\,b + m\,c + n\,d$$

linearly dependent on four noncopunctual planes a, b, c, d are all the planes of space.

It is evident from these developments that, in space, point and plane are dual and the straight line is self-dual.

We recognize in space three *fundamental forms* of points, three of planes, and four of lines:

Dimensionality	*Points*	*Planes*	*Lines*
One	Range	Pencil	Pencil
Two	Plane	Sheaf	{Plane Sheaf
Three	Space	Space	
Four			Space

The point, the plane, or the line may be taken as the fundamental element in the study of the geometry of space. We shall be interested primarily in the point and plane geometries and shall develop them simultaneously. Both of these geometries are three-dimensional and each is the dual of the other. In both, the straight line, that is, a

range of points or a pencil of planes, is one-dimensional. In plane geometry, the two-dimensional form is the point: the sheaf of planes; and, in point geometry, it is the plane: the plane of points.

We shall, on occasion, for the sake of unification and conciseness, make use of the relation between two elements known as *united position*. A point and a plane are said to have united position if the point lies in the plane, a line and a point if the line contains the point, a line and a plane if the line lies in the plane, two points or two planes if they are identical, and two lines if they intersect.

EXERCISES

1. Show that a straight line and a plane not through the line have a unique point in common. State and prove the dual.

2. Two distinct straight lines are determined respectively by the pairs of planes a, b and c, d. Establish a condition necessary and sufficient that the lines intersect. State the dual.

3. Establish the following theorems:

(*a*) Theorem 5; (*b*) Theorem 6; (*c*) Theorem 9.

4. In the following equations k, l, or k, l, m, are parameters which do not vanish simultaneously. Determine what each pair of simultaneous equations represents.

(*a*) $k(a|u) + l(b|u) = 0, \quad (c|u) = 0;$

(*b*) $k(a|x) + l(b|x) = 0, \quad (c|x) = 0;$

(*c*) $k(a|x) + l(b|x) + m(c|x) = 0, \quad (d|x) = 0;$

(*d*) $k(a|u) + l(b|u) + m(c|u) = 0, \quad (d|u) = 0.$

3. Cross Ratio. One-Dimensional Coordinate Systems. The theory of cross ratio may be extended to space by carrying over from the plane the geometric definitions of cross ratio for points and lines (Ch. VI) and introducing a similar definition for planes. We prefer, however, to develop the whole subject anew from a purely analytic point of view, based on the consideration of coordinate systems in one-dimensional fundamental forms.

One-Dimensional Coordinate Systems. Denote by E_{10}, E_{01}, E_{11} three distinct elements of a one-dimensional fundamental form, which we assume, to begin with, is a range of points or a pencil of planes. Let $a : (a_1, a_2, a_3, a_4)$ and $b : (b_1, b_2, b_3, b_4)$ be homogeneous (point or plane) coordinates for E_{10} and E_{01}, so chosen that E_{11} has the coordinates $a + b$, and let

$$E: \qquad ka + lb$$

be an arbitrary element of the form.

The ordered numbers (k, l) constitute homogeneous coordinates for the elements of the form. We shall refer to them as *projective coordinates.* The following table shows in detail their relationship to the space coordinates.

	E_{10}	E_{01}	E_{11}	E
Space Coordinates	a	b	$a + b$	$k\,a + l\,b$
Coordinates (k, l)	$(1, 0)$	$(0, 1)$	$(1, 1)$	(k, l).

The elements E_{10}, E_{01} are called the *zero elements,* and E_{11} the *unit element,* of the coordinate system.

Suppose that we set

$$a = A\,a', \qquad b = B\,b',$$

where A and B are constants, $\neq 0$, and denote the coordinates in the form which are based on a', b' as the space coordinates of E_{10}, E_{01} by (k', l'). We then have:

	E_{10}	E_{01}	E_{11}	E
Space Coordinates	a'	b'	$Aa' + Bb'$	$(Ak)a' + (Bl)b'$
Coordinates (k', l')	$(1, 0)$	$(0, 1)$	(A, B)	(Ak, Bl).

The two tables show that the coordinate systems (k, l) and (k', l'), even though they have the same zero elements, are not in general identical. They are identical only if $A = B$, that is, only if they also have the same unit element.

THEOREM 1. *If E_{10}, E_{01}, and E_{11} are three distinct elements of a one-dimensional fundamental form, there exists a unique system of projective coordinates for the form in which E_{10}, E_{01} are the zero elements* $(1, 0)$, $(0, 1)$ *and E_{11} is the unit element* $(1, 1)$.

It is well to note that the coordinate system, though not completely determined by the zero elements themselves, is completely determined by definitely chosen space coordinates a, b for the zero elements.

THEOREM 2. *The change from one coordinate system in a one-dimensional form to a second is represented by a linear transformation, and conversely.*

Let an arbitrary element E of the form, which we still assume is a range of points or a pencil of planes, have the coordinates (k, l) and (k', l') in the two given coordinate systems, and let the corresponding space coordinates of E be $k\,a + l\,b$ and $k'a' + l'b'$. Substituting the expressions for a and b as linear combinations of a', b', namely

$$a = A_1a' + B_1b', \qquad b = A_2a' + B_2b', \qquad A_1B_2 - A_2B_1 \neq 0,$$

into the symbolic equation

$$\rho(k'a' + l'b') = k\,a + l\,b, \qquad \rho \neq 0,$$

we find immediately that

$$(1) \qquad \begin{aligned} \rho\, k' &= A_1 k + A_2 l, \\ \rho\, l' &= B_1 k + B_2 l. \end{aligned}$$

Conversely, every linear transformation (1) represents a change of coordinates in the form, as is readily proved by retracing steps.

THEOREM 3. *If the basic points* (1, 0), (0, 1), (1, 1) *of a coordinate system in a range of points lie respectively in the basic planes* (1, 0), (0, 1), (1, 1) *of a coordinate system in a pencil of planes, the point* (k, l) *of the range lies in the plane* (k, l) *of the pencil.*

Let x, y, $x + y$ be the space coordinates of the basic points of the range and u, v, $u + v$, those of the basic planes of the pencil. By hypothesis,

$$(x \mid u) = 0, \qquad (y \mid v) = 0, \qquad (x + y \mid u + v) = 0,$$

or

$$(x \mid u) = 0, \qquad (y \mid v) = 0, \qquad (x \mid v) + (y \mid u) = 0.$$

Hence

$$(k\,x + l\,y \mid k\,u + l\,v) \equiv k^2(x \mid u) + kl\,[(x \mid v) + (y \mid u)] + l^2(y \mid v) \equiv 0.$$

Consider now a pencil of lines. Choose arbitrarily a range of points whose points lie respectively on the lines of the pencil, and a pencil of planes whose planes contain respectively the lines of the pencil (Fig. 1). Let L_{10}, L_{01}, L_{11} be three distinct fixed lines and L an arbitrary line of the pencil, P_{10}, P_{01}, P_{11}, and P the points of the range which lie on these lines, and M_{10}, M_{01}, M_{11}, and M the planes of the pencil of planes which contain them. Construct the system of coordinates in the range of points based on P_{10}, P_{01}, P_{11} and that in the pencil of planes based on M_{10}, M_{01}, M_{11}. The point P of the range and the plane M of the pencil have then, by Th. 3, the same coordinates (k, l). These coordinates we assign to the line L. We thus obtain a coordinate

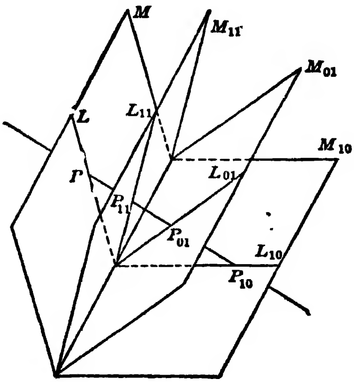


FIG. 1

system in the pencil of lines in which L_{10}, L_{01} are the zero lines (1, 0), (0, 1) and L_{11} is the unit line (1, 1).

That this coordinate system is independent of the particular range of points and the particular pencil of planes originally chosen is readily proved by applying Theorem 3 to two positions of the range and one position of the pencil, or to two positions of the pencil and one position of the range. Hence, Theorem 1 holds for a pencil of lines.

A pencil of lines may be thought of as consisting of the lines which join a point $(c|u) = 0$ to the points of a range of points $(k\,a + l\,b|u) = 0$ not containing the point c, or as consisting of the lines of intersection of a plane $(c'|x) = 0$ with the planes of a pencil of planes $(k'a' + l'b'|x) = 0$ not containing the plane c'. Thus, it is represented analytically by either of the pairs of equations

(2) $(k\,a + l\,b|u) = 0, \quad (c|u) = 0; \qquad (k'a' + l'b'|x) = 0, \quad (c'|x) = 0.$

The coordinates (k, l) in the range of points and the coordinates (k', l') in the pencil of planes are then coordinates in the pencil of lines and, in fact, the same coordinates if the basic points of the range of points lie in the basic planes of the pencil of planes.

The manner of construction of the coordinate systems in a pencil of lines guarantees the truth of Theorem 2 for a pencil of lines and shows that Theorem 3 is valid also for a range of points and a pencil of lines and for a pencil of lines and a pencil of planes. The extension of Theorem 3 we state in terms of the concept of united position.

THEOREM 4. *If the basic elements of a coordinate system in a one-dimensional form have respectively united position with the corresponding basic elements of a coordinate system in a one-dimensional form of a different type, the element (k, l) of the first form has united position with the element (k, l) of the second form.*

If (k, l) are homogeneous coordinates in a one-dimensional form,

$$w = \frac{k}{l}$$

is defined as the corresponding *nonhomogeneous coordinate.* Every element of the form, except E_{10}, has a unique coordinate w. In particular, w becomes infinite for E_{10}, $w = 0$ for E_{01}, and $w = 1$ for E_{11}.

Cross Ratio. If, of four distinct elements in a one-dimensional form, three are chosen as the basic elements of a coordinate system in the form, the fourth takes on a *definite* nonhomogeneous coordinate (Th. 1).

Thus, there are associated with the four elements certain numbers. We call these numbers the cross ratios of the four elements.

DEFINITION. *If E_1, E_2, E_3, E_4 are four distinct elements of a one-dimensional fundamental form, the cross ratio (E_1E_2, E_3E_4) is the non-homogeneous coordinate w of E_4 in the coordinate system in which E_1, E_2, E_3 are respectively the basic elements E_{10}, E_{01}, E_{11}:*

$$(E_1E_2, E_3E_4) = w. \tag{3}$$

It is clear from the definition that $w \neq 0, 1$, and that, when the first three elements and the value of $w(\neq 0, 1)$ are given, the fourth element is uniquely determined.

THEOREM 5. *If four distinct elements of a one-dimensional form are respectively in united position with four distinct elements of a one-dimensional form of different type, a cross ratio of the four elements of the first form is equal to the corresponding cross ratio of the four elements of the second form.*

The theorem is essentially a restatement of Theorem 4 in terms of cross ratio.

THEOREM 6. *If E_1, E_2, E_3, E_4 have the homogeneous coordinates (k_i, l_i), $i = 1, 2, 3, 4$, in an arbitrarily chosen, but fixed, coordinate system in the form,*

$$(E_1E_2, E_3E_4) = \frac{|k_3\, l_1|\,|k_4\, l_2|}{|k_3\, l_2|\,|k_4\, l_1|}. \tag{4}$$

It suffices to prove the proposition for the case of a range of points or a pencil of planes. Let the space coordinates of E_1, E_2, E_3, E_4 which give rise respectively to the coordinates w and (k, l) in the form be

$$\begin{array}{cccc} a & b & a+b & w\,a+b, \\ k_1a'+l_1b' & k_2a'+l_2b' & k_3a'+l_3b' & k_4a'+l_4b'. \end{array}$$

Then

$$\begin{aligned} k_1a' + l_1b' &= \rho_1 a, & k_3a' + l_3b' &= \rho_3(a+b), \\ k_2a' + l_2b' &= \rho_2 b, & k_4a' + l_4b' &= \rho_4(w\,a+b). \end{aligned}$$

Solving the two equations on the left for a', b' in terms of a, b, we obtain

$$|k_1\, l_2|\,a' = \rho_1 l_2 a - \rho_2 l_1 b, \qquad |k_1\, l_2|\,b' = -\rho_1 k_2 a + \rho_2 k_1 b.$$

Substituting these expressions for a', b' in the two equations on the right, we arrive at the following relations

$$\begin{aligned} \rho_1|k_3\, l_2| &= \rho_3|k_1\, l_2|, & \rho_1|k_4\, l_2| &= w\,\rho_4|k_1\, l_2|, \\ -\rho_2|k_3\, l_1| &= \rho_3|k_1\, l_2|, & -\rho_2|k_4\, l_1| &= \rho_4|k_1\, l_2|. \end{aligned}$$

Hence

$$-\frac{\rho_1}{\rho_2} = \frac{|k_3 l_1|}{|k_3 l_2|}, \qquad -\frac{\rho_1}{\rho_2}\frac{|k_4 l_2|}{|k_4 l_1|} = w,$$

and *

(5) $$w = \frac{|k_3 l_1|\,|k_4 l_2|}{|k_3 l_2|\,|k_4 l_1|}.$$

The formulas for cross ratio given in Ch. VI, § 4 are valid for points of a range (planes of a pencil) provided the symbols a, b are thought of as representing space coordinates of points (planes). The same formulas hold also for lines of a pencil in space when the symbols are interpreted as the a, b or the a', b' of the first or second pair of equations (2).

Projective correspondences between two one-dimensional forms in space may now be treated in precisely the same way as they were in the plane. In fact, the definition, theorems, and proofs of Ch. IX, § 4 are equally valid here. The only difference is that we now have three kinds of forms instead of two.

An Analytic Foundation for Projective Geometry. From the point of view of *pure* projective geometry, the foregoing theory of cross ratio has the advantage that it is not directly dependent on the metric concept of distance. But, like the rest of our present development of projective geometry, it does depend indirectly on metric concepts in that it is based on Cartesian point and plane coordinates.

We digress for a moment to outline a method by means of which it is possible to free our present treatment completely from metric elements and at the same time to establish it on a firmer foundation.

We assume two classes of objects which we call points and planes and lay down for them the following postulates.

POSTULATE 1 *a.* *To each point corresponds a set of ordered number quadruples* $(\rho x_1, \rho x_2, \rho x_3, \rho x_4)$ *where* x_1, x_2, x_3, x_4 *are fixed, not all zero, and* ρ *is arbitrary, not zero, and conversely.*

POSTULATE 1 *b.* *To each plane corresponds a set of ordered number quadruples* $(\rho u_1, \rho u_2, \rho u_3, \rho u_4)$ *where* u_1, u_2, u_3, u_4 *are fixed, not all zero, and* ρ *is arbitrary, not zero, and conversely.*

* The proof amounts to showing that the expression on the right-hand side of (4) or (5) is an absolute invariant of four distinct elements of the form with respect to the linear transformations which represent the changes of coordinates (see Th. 2). If we had cared to make use of our previous knowledge of this fact, reference to it would have been sufficient proof of our theorem.

DEFINITION 1 *a*. A number quadruple (x_1, x_2, x_3, x_4) shall be called a set of homogeneous coordinates for the corresponding point.

DEFINITION 1 *b*. A number quadruple (u_1, u_2, u_3, u_4) shall be called a set of homogeneous coordinates for the corresponding plane.

We next define the fundamental relationship between point and plane.

DEFINITION 2. *If coordinates (x_1, x_2, x_3, x_4) of a point and coordinates (u_1, u_2, u_3, u_4) of a plane satisfy the condition*

$$u_1x_1 + u_2x_2 + u_3x_3 + u_4x_4 = 0,$$

the point shall be said to lie in the plane and the plane to contain the point.

The reader may now prove

THEOREM 1. *There exist configurations each consisting of a subclass (A) of ∞^1 points and a subclass (B) of ∞^1 planes, which are so related that the points of (A) lie in every plane of (B) and are all the points lying in every plane of (B), and the planes of (B) contain every point of (A) and are all the planes containing every point of (A).*

DEFINITION 3. *These configurations shall be called straight lines.*

This completes the essentials of an analytic foundation on which our present development of projective geometry, beginning with § 2, may be based.

EXERCISES

1. Two projective pencils of planes are said to be *perspective* if the lines of intersection of corresponding planes are coplanar. Prove that a necessary and sufficient condition that two projective pencils of planes be perspective is that their axes be coplanar and their common plane self-corresponding, and show that the lines of intersection of corresponding planes form, then, a pencil of lines. State the duals of the definition and theorem.

2. The axes of two projective, nonperspective pencils of planes are coplanar. Discuss in full the locus of the lines of intersection of corresponding planes of the two pencils.

3. Define and develop two-dimensional coordinate systems, first for a plane of points and a sheaf of planes, and then for a sheaf of lines and a plane of lines.

4. Show that the coordinate systems for a plane of points and a plane of lines are identical with those of Ch. X.

5. Define and discuss projective correspondences between two-dimensional fundamental forms.

6. Two projective planes of points are said to be *perspective* if the lines joining corresponding points are concurrent. Show that a necessary and sufficient

condition that two projective planes of points be perspective is that each point common to the two planes be self-corresponding.

7. State the duals of the definition and theorem of the previous exercise and prove the new theorem.

8. Prove Theorem 1 of the abstract development of projective geometry.

4. Projective Geometry at a Point. Quadric Cones. The projective geometry at a point P is the projective geometry of the sheaf of lines and the sheaf of planes which have P as vertex. It is the composite of a line geometry at P, in which the plane appears as a pencil of lines, and a plane geometry at P, in which the line appears as a pencil of planes.

The two geometries may be developed simultaneously in standard analytic forms by choosing projective coordinates (ξ_1, ξ_2, ξ_3) in the sheaf of lines and projective coordinates (η_1, η_2, η_3) in the sheaf of planes (§ 3, Ex. 3) so that the line ξ lies in the plane η when and only when $(\xi|\eta) = 0$. The equation $(a|\xi) = 0$ is, then, the equation in line coordinates of the plane a, and the equation $(a|\eta) = 0$ is the equation in plane coordinates of the line a.

The quadratic equation in line coordinates

$$(1\,a) \quad \sum_{ij}^{1-3} a_{ij}\xi_i\xi_j = 0, \quad a_{ij} = a_{ji},$$

is said to represent a *quadric cone of lines*, nondegenerate or degenerate according as $|a_{ij}| \neq 0$ or $|a_{ij}| = 0$.

A degenerate cone of lines consists of two planes (pencils of lines), distinct or coincident.

The quadratic equation in plane coordinates

$$(1\,b) \quad \sum_{ij}^{1-3} b_{ij}\eta_i\eta_j = 0, \quad b_{ij} = b_{ji},$$

is said to represent a *quadric cone of planes*, nondegenerate or degenerate according as $|b_{ij}| \neq 0$ or $|b_{ij}| = 0$.

A degenerate cone of planes consists of two lines (pencils of planes), distinct or coincident.

The configurations obtained by adjoining to the nondegenerate cones of lines their "tangent planes" and those obtained by adjoining to the nondegenerate cones of planes their "contact lines" are identical. We call them *nondegenerate cones*.

In justification of these statements, think of the coördinates ξ and η of an arbitrary line L and an arbitrary plane p as assigned also to the point and line in which L and p meet a fixed plane not containing P. Equations (1 a) and (1 b) then represent not only the cone of lines and the cone of planes under investigation, but also the point conic and

line conic in which these cones intersect the fixed plane. Hence the analytic theories of the cones are identical with those of the conics and the geometric results differ only in terminology.

EXERCISES

1. Develop the theory of contact lines of a nondegenerate cone of planes.

2. Discuss in detail the envelope of the planes determined by corresponding lines of two projective pencils of lines which have the same vertex, but lie in different planes.

3. Find the locus of a line which always contains a fixed point P and varies so that the cross ratio of the four planes determined by it and four fixed lines through P is constant.

4. Show that there is a unique cone of planes, determined by five copunctual planes, no three coaxal, and that the cone is nondegenerate. State the dual.

5. Develop the polar system associated with a nondegenerate cone.

6. Prove that the degenerate point conics and the degenerate cones of planes are essentially identical.

7. What is the space dual of a line conic?

8. Show that the cone of lines, considered in space, has ∞^2 points and hence may be represented by a single equation in the point coordinates of space. State the space dual.

9. Can a point conic ever be represented by a single equation in the plane coordinates of space? By a single equation in the point coordinates of space?

5. Three-Dimensional Coordinate Systems. In a three-dimensional fundamental form, a space of points or a space of planes, let E_1, E_2, E_3, E_4 be four linearly independent elements and let E be an arbitrary element. If a, b, c, d, and z represent homogeneous coordinates of E_1, E_2, E_3, E_4, and E, homogeneous constants z'_1, z'_2, z'_3, z'_4, and ρ exist so that

$$(1) \qquad \rho z = z'_1 a + z'_2 b + z'_3 c + z'_4 d, \qquad \rho \neq 0.$$

DEFINITION. *The ordered number set* (z'_1, z'_2, z'_3, z'_4) *shall constitute new homogeneous coordinates of the arbitrary element* E.

The elements E_1, E_2, E_3, E_4 have the new coordinates (1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0), (0, 0, 0, 1). They are called the *zero elements*, and the tetrahedron which they determine, *the tetrahedron of reference*, of the new coordinate system.

Since the original coordinates a, b, c, d of the zero elements admit factors of proportionality which affect (z'_1, z'_2, z'_3, z'_4), the zero elements do not completely determine the new coordinate system. To render

the new coordinates unique it is necessary, and sufficient, to choose a fifth element E_u *linearly independent of each three of the zero elements,* and to prescribe for it definite new coordinates, linearly independent of those of each three of the zero elements. In particular, we prescribe $(1, 1, 1, 1)$ as coordinates for E_u, and accordingly call E_u the *unit element.*

THEOREM 1. *If E_1, E_2, E_3, E_4, and E_u are five elements of a three-dimensional fundamental form, no four of which are linearly dependent, there exists a unique system of new homogeneous coordinates in the form in which E_1, E_2, E_3, E_4 are the zero elements* $(1, 0, 0, 0)$, $(0, 1, 0, 0)$, $(0, 0, 1, 0)$, $(0, 0, 0, 1)$, *and E_u is the unit element* $(1, 1, 1, 1)$.

The proof of this and the following theorem we leave to the reader.

THEOREM 2. *Homogeneous Cartesian coordinates are a special case of the new coordinates.*

Equations (1),

$$\rho z_i = a_i z_1' + b_i z_2' + c_i z_3' + d_i z_4', \qquad (i = 1, 2, 3, 4),$$

represent the transformation from the new coordinates z' to the original coordinates z. The transformation is linear and, since $|a\,b\,c\,d| \neq 0$, the inverse is linear, of the form

$$\text{(2)} \qquad \sigma z_i' = \sum_{j=1}^{4} a_{ij} z_j, \quad (i = 1, 2, 3, 4), \quad |a_{ij}| \neq 0.$$

Hence we readily prove

THEOREM 3. *The change from one system of homogeneous coordinates in a three-dimensional fundamental form to a second is represented by a (nonsingular) linear transformation, and conversely.*

As a consequence of this theorem, we have:

THEOREM 4. *The projective geometry thus far developed for a space of points (planes) has the same analytical form in terms of the new point (plane) coordinates as in terms of the original point (plane) coordinates.*

In § 3 we introduced in a one- (two-)dimensional form one- (two-) dimensional coordinate systems based on the original space coordinates. We may now introduce in the form similarly constructed coordinate systems based on the new space coordinates. How will these new coordinate systems be related to the original ones?

Consider the question, say, for a range of points. Let P_{10}, P_{01}, P_{11} be the basic points and P an arbitrary point of the range, and let (k, l) be the coordinates in the range based on the original space coordinates

a, b, $a + b$, and $k\,a + l\,b$ for P_{10}, P_{01}, P_{11}, and P. New space coordinates for the four points, obtained from a, b, $a + b$, $k\,a + l\,b$ by means of (2), are a', b', $\sigma_1 a' + \sigma_2 b'$, $k\,\sigma_1 a' + l\,\sigma_2 b'$, where σ_1, σ_2 are the values of σ for $a \to a'$, $b \to b'$. In order that P_{11} be the unit point for the new coordinates $(k\,\sigma_1, l\,\sigma_2)$ in the range, a' and b' must be so chosen that $\sigma_1 = \sigma_2$. The new coordinates in the range are then identical with the original coordinates (k, l).

THEOREM 5. *The one- (two-) dimensional coordinate systems in a one- (two-) dimensional form are independent of the space coordinates, that is, are invariant with respect to a change of space coordinates.*

This result is most fortunate in that it guarantees that our definition of cross ratio (§ 3) is independent of a change of space coordinates.

Point and Plane Coordinates in the Same Space. Let x' and u' be new point and plane coordinates in space, defined in terms of the original point and plane coordinates x and u by the equations

$$\text{(3)} \qquad \rho x_i = \sum_{j=1}^{4} a_{ij} x'_j, \qquad \sigma u_i = \sum_{k=1}^{4} b_{ik} u'_k, \quad (i = 1, 2, 3, 4).$$

Then

$$\rho\sigma \sum_{i=1}^{4} x_i u_i = \sum_{i=1}^{4} \sum_{j,k}^{1-4} a_{ij} b_{ik} x'_j u'_k = \sum_{j,k}^{1-4} \left(\sum_{i=1}^{4} a_{ij} b_{ik} \right) x'_j u'_k,$$

or

$$\rho\sigma(x \,|\, u) = \sum_{j,k}^{1-4} c_{jk} x'_j u'_k, \quad \text{where} \quad c_{jk} = \sum_{i=1}^{4} a_{ij} b_{ik}.$$

LEMMA. *If x' and u' are new point and plane coordinates related to the original ones by equations* (3), *the condition necessary and sufficient that the point x' lie in the plane u' is*

$$\text{(4)} \qquad \sum_{j,k}^{1-4} c_{jk} x'_j u'_k = 0, \quad \textit{where} \quad c_{jk} = \sum_{i=1}^{4} a_{ij} b_{ik}.$$

This condition reduces to the standard form: $(x' \,|\, u') = 0$ if and only if

$$\text{(5)} \qquad c_{jk} = 0, \qquad c_{jj} = c_{kk} \neq 0, \qquad (j \neq k, \quad j, k = 1, 2, 3, 4).$$

The twelve equations $c_{jk} = 0$ may be arranged in four sets of three each, corresponding to the four values of k. The typical set is

$$\begin{aligned} a_{1p} b_{1s} + a_{2p} b_{2s} + a_{3p} b_{3s} + a_{4p} b_{4s} &= 0, \\ a_{1q} b_{1s} + a_{2q} b_{2s} + a_{3q} b_{3s} + a_{4q} b_{4s} &= 0, \\ a_{1r} b_{1s} + a_{2r} b_{2s} + a_{3r} b_{3s} + a_{4r} b_{4s} &= 0, \end{aligned}$$

where p, q, r, s is a cyclic order of 1, 2, 3, 4. Hence

$$b_{1s} : b_{2s} : b_{3s} : b_{4s} = A_{1s} : A_{2s} : A_{3s} : A_{4s}, \quad (s = 1, 2, 3, 4),$$

or

$$b_{ij} = \lambda_j A_{ij}, \qquad (i, j = 1, 2, 3, 4),$$

where A_{ij} is the cofactor of a_{ij} in the determinant $|a_{ij}|$ and the λ_j's are factors of proportionality. Now

$$c_{jj} = \sum_{i=1}^{4} a_{ij} b_{ij} = \lambda_j \sum_{i=1}^{4} a_{ij} A_{ij} = |a_{ij}| \lambda_j,$$

so that $c_{jj} = c_{kk}$, $j, k = 1, 2, 3, 4$, if and only if $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4$. Consequently, equivalent to equations (5) are the relations

$$(6\,a) \qquad b_{ij} = \lambda\, A_{ij}, \quad \lambda \neq 0, \quad (i, j = 1, 2, 3, 4),$$

or, since equations (5) are symmetric in the a's and b's, the relations

$$(6\,b) \qquad a_{ij} = \mu\, B_{ij}, \quad \mu \neq 0, \quad (i, j = 1, 2, 3, 4).$$

THEOREM 6. *A necessary and sufficient condition that the transformations* (3) *from the original coordinates x, u to the new coordinates x', u' preserve the standard form of the condition that a point lie in a plane is that*

$$(6) \qquad b_{ij} = \lambda\, A_{ij}, \quad \textit{or} \quad a_{ij} = \mu\, B_{ij}, \quad \lambda\mu \neq 0, \quad (i, j = 1, 2, 3, 4).$$

Point and plane coordinates for which the condition that a point lie in a plane is in standard form we shall call *associated* point and plane coordinates. It is clear from Theorem 6 that, associated with a given system of point coordinates, is a unique system of plane coordinates, and vice versa.

The transformations

$$(7\,a) \qquad \rho x_i = \sum_{j=1}^{4} a_{ij} x'_j, \qquad \sigma u_i = \sum_{j=1}^{4} A_{ij} u'_j, \quad (i = 1, 2, 3, 4),$$

or the inverse transformations

$$(7\,b) \qquad \rho' x'_i = \sum_{j=1}^{4} A_{ji} x_j, \qquad \sigma' u'_i = \sum_{j=1}^{4} a_{ji} u_j, \quad (i = 1, 2, 3, 4),$$

represent changes from associated point and plane coordinates x, u to new associated point and plane coordinates x', u'.

Geometrical Interpretations. Let $A_1 : (1, 0, 0, 0)$, $A_2 : (0, 1, 0, 0)$, $A_3 : (0, 0, 1, 0)$, $A_4 : (0, 0, 0, 1)$ be the vertices of the tetrahedron of reference and $D : (1, 1, 1, 1)$ the unit point of a system of point coordinates. Denote by a_i the face of the tetrahedron opposite to A_i,

by E_{ij} the edge of the tetrahedron common to the faces a_i, a_j, and by d_{ij} and p_{ij} the planes determined by the edge E_{ij} and the points D and P, where P is an arbitrary point.

THEOREM 7. *If (x_1, x_2, x_3, x_4) are coordinates of P, the ratio x_j/x_i is equal to the cross ratio in which the faces a_i, a_j are separated by the planes d_{ij}, p_{ij}:*

$$\frac{x_j}{x_i} = \lambda_{ij}, \quad \textit{where} \quad \lambda_{ij} = (a_i a_j, d_{ij} p_{ij}), \quad (i \neq j, \quad i, j = 1, 2, 3, 4),$$

and conversely.

The proof is left to the reader (Ex. 4). Let him show, also, that if P is restricted to the face $a_i : x_i = 0$, then (x_j, x_k, x_l) are the point coordinates for the face a_i which are based on the triangle $A_j A_k A_l$ as triangle of reference and the point D_i in which the line $A_i D$ meets a_i as unit point. The points D_i are known as the unit points of the faces a_i.

The projective interpretation of plane coordinates is the dual of that of point coordinates. Here, it is the faces a_i of a tetrahedron and a plane d which are respectively the zero elements and the unit element. If, in particular, the variable plane $p : (u_1, u_2, u_3, u_4)$ is restricted to pass through the vertex $A_i : u_i = 0$, then (u_j, u_k, u_l) are the line coordinates for the lines of the plane a_i which are based on the triangle $A_j A_k A_l$ as triangle of reference and the line d_i in which d meets a_i as unit line. The lines d_i are called the unit lines in the faces a_i.

THEOREM 8. *A necessary and sufficient condition that point and plane coordinates in the same space be associated is that the zero points A_i and the zero planes a_i be respectively the vertices and opposite faces of a tetrahedron, and that the unit point and unit line in each face bear the same relationship to one another with respect to the triangle in the face as in the case of associated point and line coordinates in a plane.*

Our point and plane coordinates, thus interpreted, are known as *projective coordinates*. They are also capable of metrical interpretations whereby they appear as the generalizations of trilinear coordinates in the plane. They are then called *tetrahedral coordinates.*

EXERCISES

1. Prove Theorem 1. Why is it necessary that E_u be linearly independent of each three of the zero elements?

2. Establish Theorem 2. Describe accurately the common tetrahedron of reference for associated Cartesian point and plane coordinates.

3. Prove Theorem 5 in the case of a sheaf of planes.

4. Prove Theorem 7. Show also that

$$\lambda_{ij}\lambda_{ji} = 1, \qquad (i, j = 1, 2, 3, 4),$$
$$\lambda_{34}\lambda_{42}\lambda_{23} = 1, \qquad \lambda_{12}\lambda_{24}\lambda_{41} = 1,$$
$$\lambda_{41}\lambda_{13}\lambda_{34} = 1, \qquad \lambda_{23}\lambda_{31}\lambda_{12} = 1.$$

5. Establish conditions necessary and sufficient that six planes, which pass one through each edge of a tetrahedron and are distinct from the faces, be copunctual.

6. Discuss in detail the projective interpretation of plane coordinates. State the dual of the theorem established in the previous exercise.

7. Prove Theorem 8.

8. *Extension of Desargues' Triangle Theorem to Tetrahedra.* If the lines joining corresponding vertices of two tetrahedra are concurrent, the four lines of intersection of corresponding faces are coplanar and corresponding edges intersect in the vertices of the complete quadrilateral determined by these four lines.

9. State and prove the dual theorem.

10. Corresponding edges of two tetrahedra intersect. What follows?

11. Show that the lines joining corresponding vertices of the common tetrahedron of reference $A_1A_2A_3A_4$ for associated point and plane coordinates and the tetrahedron $D_1D_2D_3D_4$ are concurrent in the unit point D and the lines of intersection of corresponding faces lie in the unit plane d. State the dual.

12. Show that, if six distinct points, one on each edge of a tetrahedron, are coplanar, the six planes determined by their harmonic conjugates, with respect to the vertices, and the opposite edges are copunctual.

13. Show that the six planes determined by the edges of a tetrahedron and the mid-points of the opposite edges are copunctual.

14. Discuss in detail tetrahedral point coordinates.

6. Three-Dimensional Projective Transformations. Two spaces of points are projective or in projective correspondence if (*a*) to a point of the one corresponds a point of the other, (*b*) to the points of a plane in the one correspond the points of a plane in the other, (*c*) to the points of a line in the one correspond the points of a line in the other, and (*d*) cross ratio is preserved.

Only one of the requirements (*b*), (*c*) is necessary, since, if either is satisfied, so is the other. If to a range of points corresponds a range of points, then to the points of a plane correspond the points of a plane, inasmuch as a plane of points may be thought of as generated by the range of points determined by a point tracing a fixed range and a fixed point not contained in this range. Conversely, if to a plane corresponds a plane, then to a line corresponds a line.

Analogous remarks apply to the case of two spaces of planes or to that of a space of points and a space of planes. We may, then, formulate our general definition as follows.

DEFINITION. *Two three-dimensional fundamental forms whose elements are in one-to-one correspondence so that to the elements of a two- (one-) dimensional form of the one correspond the elements of a two- (one-) dimensional form of the other and cross ratio is preserved, are projective or in projective correspondence.*

THEOREM 1. *There is a unique projective correspondence between two three-dimensional forms which orders respectively to five given elements of the one form five prescribed elements of the other, provided only that no four elements of either set of five are linearly dependent.*

Take the five given elements as the basic elements for a coordinate system in the first form and the five prescribed elements as the corresponding basic elements of a coordinate system in the second form. If there exists a projective correspondence determined by the five pairs of elements, corresponding elements must have the same coordinates. But the correspondence

$$(1) \qquad \rho z_i' = z_i, \qquad (i = 1, 2, 3, 4),$$

is surely projective.

The transformations determined by projective correspondences we call, as usual, projective transformations.

THEOREM 2. *Every projective transformation of one three-dimensional form into a second, expressed in terms of arbitrarily chosen coordinates in the two forms, is a linear transformation, and conversely.*

For, on the one hand, (1) is linear and a change of coordinates is linear, and on the other hand, every linear transformation of one space into a second has the properties characteristic of a projective transformation.

Collineations and Correlations. These terms are employed as in the plane. The collineations of space are the projective transformations of space into itself which carry points into points and hence planes into planes and lines into lines, or the projective transformations which carry planes into planes and hence points into points and lines into lines. If a collineation and its inverse have as their equations in point coordinates

$$(2\,a) \quad \rho x_i' = \sum_{j=1}^{4} a_{ij} x_j, \qquad \sigma x_i = \sum_{j=1}^{4} A_{ji} x_j', \quad (i = 1, 2, 3, 4), \quad |a_{ij}| \neq 0,$$

then their equations in the associated plane coordinates are

$$(2\,b)\qquad \lambda u_i' = \sum_{j=1}^{4} A_{ij} u_j, \qquad \mu u_i = \sum_{j=1}^{4} a_{ji} u_j', \quad (i = 1, 2, 3, 4).$$

THEOREM 3. *The collineations of space constitute a fifteen-parameter group.*

The correlations of space are the projective transformations of space into itself which carry points into planes and hence planes into points and lines into lines, or the projective transformations which carry planes into points and hence points into planes and lines into lines.

THEOREM 4. *The $2\cdot\infty^{15}$ collineations and correlations constitute the group of all projective transformations of space.*

It is to be noted that both a collineation and a correlation carry a line into a line.

EXERCISES

1. Enlarge on the proofs of Theorems 1 and 2.

2. Deduce equations (2 b) from equations (2 a), first directly, and then by applying § 5, Th. 6.

3. *Homologies.* A point transformation of space leaves fixed a point A and each point of a plane a, not containing A, and carries every other point P into the point P' on the line AP for which $(A\,P_0,\,P\,P') = k$, where P_0 is the point in which AP meets a, and k is a constant, $\neq 0, 1$. Show that the transformation is a collineation and discuss its effect on the planes and lines of space. Find a canonical form for its equations.

The transformation is known as a *homology*, the point A as its *center*, the plane a as its *central plane*, and k as its *invariant*.

Prove that a homology is involutory if and only if its invariant is equal to -1.

4. *Skew Involutions.* A point transformation of space leaves every point on each of two skew lines L_1, L_2 fixed and carries every other point P into its harmonic conjugate with respect to the two points on L_1 and L_2 which are collinear with P. Show that the transformation is an involutory collineation, and discuss its effect on planes and lines. Find a canonical form for its equations.

The transformation is called a *skew involution*, and the lines L_1, L_2, its *axes*.

5. Show that a reflection in a point and a reflection in a plane are metric examples of involutory homologies, and that reflection in a line is a metric example of a skew involution. Prove that, if a point lies in a plane, the product of the reflection in the point and the reflection in the plane is a reflection in a line.

6. The center of each of two involutory homologies lies in the central plane of the other. Prove that their product, in either order, is the same skew involution.

7. Show that an involutory collineation is either an involutory homology or a skew involution.

8. If

$$\rho u_i' = \sum_{j=1}^{4} a_{ij} x_j, \qquad |a_{ij}| \neq 0, \quad (i = 1, 2, 3, 4),$$

are the equations of a correlation as a transformation of points into planes, show that

$$\rho x_i' = \sum_{j=1}^{4} A_{ij} u_j, \qquad (i = 1, 2, 3, 4),$$

are the equations of the same correlation as a transformation of planes into points. Prove that the correlation may be equally well represented by either of the bilinear equations

$$\sum_{i,j}^{1-4} a_{ij} x_i' x_j = 0, \qquad \sum_{i,j}^{1-4} A_{ij} u_i' u_j = 0.$$

7. Metric Geometry. We return to the extended Cartesian space of § 1, which we now assume to be a complex space,* and discuss its metric or Euclidean geometry.

Isotropic Directions and Lines. The square of the distance between two finite points on the line represented parametrically by the equations

$$x = x_0 + l\,t, \qquad y = y_0 + m\,t, \qquad z = z_0 + n\,t$$

is

$$D^2 = (t_2 - t_1)^2(l^2 + m^2 + n^2),$$

where t_1, t_2 are the values of t for the two points.

Definition. *A finite line, or direction, with direction components l, m, n is isotropic if*

$$l^2 + m^2 + n^2 = 0.$$

Theorem 1. *A finite line is an isotropic line if and only if the distance between each two finite points on it is zero.*

The point at infinity in the direction with components x_1, x_2, x_3 has the coordinates $(x_1, x_2, x_3, 0)$. Hence:

Theorem 2. *The locus of the points at infinity in the isotropic directions is the conic represented by the equations*

$$x_1^2 + x_2^2 + x_3^2 = 0, \qquad x_4 = 0. \tag{1}$$

* Up to this point there has been need to commit ourselves definitely to either the real or the complex domain only in the study of quadric cones. With this exception, the fundamentals of projective geometry treated in the previous paragraphs are the same in both domains and may be thought of as having been developed in either domain.

This conic is known as the *circle at infinity*, or as the *absolute conic*, or simply as the *absolute*.

THEOREM 3. *A necessary and sufficient condition that a quadric be a sphere is that it intersect the plane at infinity in the circle at infinity.*

The proof of the theorem we leave to the reader. We note a special case of interest, that of a null sphere. If the center of the sphere is, say, at the origin, the equation of the sphere is

$$x^2 + y^2 + z^2 = 0, \qquad \text{or} \qquad x_1^2 + x_2^2 + x_3^2 = 0.$$

But this is the equation of a quadric cone, namely, the cone whose rulings are the isotropic lines through the origin. Accordingly, a null sphere is frequently called *an isotropic cone*.

THEOREM 4. *The isotropic lines through a finite point constitute a quadric cone.*

An isotropic direction has no direction cosines. Direction components are, in particular, direction cosines only when the sum of their squares is equal to unity, whereas the sum of the squares of direction components of an isotropic direction is always zero.*

Isotropic Planes. A finite plane which is tangent to the absolute, that is, which meets the plane at infinity in a line tangent to the absolute, is known as an *isotropic* or *minimal* plane.

THEOREM 5. *A necessary and sufficient condition that the finite plane*

$$(2) \qquad a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4 = 0, \qquad a_1,\ a_2,\ a_3 \textit{ not all zero,}$$

be an isotropic plane is that

$$(3) \qquad a_1^2 + a_2^2 + a_3^2 = 0.$$

Its point of contact with the absolute is then $(a_1, a_2, a_3, 0)$.

For, the line at infinity in the plane (2), namely

$$a_1x_1 + a_2x_2 + a_3x_3 = 0, \qquad x_4 = 0,$$

is tangent to the absolute (1) if and only if its pole $(a_1, a_2, a_3, 0)$ with respect to the absolute lies on the absolute.

* Direction cosines of an oriented or directed line may be defined either as the cosines of the angles which the oriented line makes with the positive axes or as the projections on the axes of a segment on the line which is of unit length and directed in the same sense as the line. For an isotropic line both definitions fail, the first because the angles in question do not exist, and the second for lack of line-segments of lengths other than zero. On the other hand, direction components, defined as the projections on the axes of a directed line-segment on the line, exist as well for isotropic as for nonisotropic lines.

The line at infinity in an isotropic plane contains just one point of the circle at infinity. In other words:

THEOREM 6. *An isotropic plane contains only one pencil of isotropic lines.*

The isotropic planes through a finite point are the planes determined by the point and the tangents to the absolute, and therefore are the tangent planes of the cone formed by the lines joining the point to the points of the absolute.

THEOREM 7. *The isotropic planes through a finite point are the tangent planes to the isotropic cone whose vertex is the point.*

The lines normal to the finite plane (2) have the direction components a_1, a_2, a_3. If (2) is an isotropic plane, these lines are the isotropic lines through the circular point at infinity in the isotropic plane, and therefore are actually parallel to or lie in the isotropic plane. Strictly speaking, then, there are no lines normal to an isotropic plane.

The Projective Interpretations of Perpendicularity and Angle. The reader will find no difficulty in establishing the following facts.

THEOREM 8. *Two finite nonisotropic lines (planes) are mutually perpendicular if and only if their points (lines) at infinity are conjugate with respect to the absolute. A necessary and sufficient condition that a finite nonisotropic line and a finite nonisotropic plane be perpendicular is that the point at infinity on the line and the line at infinity in the plane be pole and polar with respect to the absolute. Three finite nonisotropic lines (planes) are mutually perpendicular if and only if their points (lines) at infinity are the vertices (sides) of a triangle which is self-conjugate with respect to the absolute.*

Laguerre's projective interpretation of angle (Ch. VIII, § 6) is valid in space as well as in the plane (Ex. 3).

Rectangular Trihedrals. If $a : a_1, a_2, a_3$; $b : b_1, b_2, b_3$; $c : c_1, c_2, c_3$ are sets of direction components of three *ordered* real lines through a finite point P, the relations

$$(4) \qquad (a|a) = (b|b) = (c|c) = 1, \qquad (a|b) = (b|c) = (c|a) = 0,$$

constitute necessary and sufficient conditions that the lines be *oriented* and *mutually perpendicular*. The three lines shall then be said to form a *rectangular trihedral.*

There are two types of rectangular trihedrals, the right-handed

(Fig. 2 a), and the left-handed (Fig. 2 b). Two trihedrals of the same type are equivalent with respect to the group of rigid motions, whereas a reflection in a point, or a plane, must be adjoined to a rigid motion in order to carry one of two trihedrals of different types into the other.

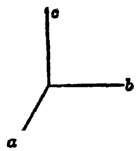


Fig. 2 a

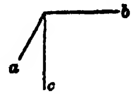


Fig. 2 b

A direct consequence of conditions (4), obtained by multiplying the determinant $|a\,b\,c|$ by itself, is the relation

$$|a\,b\,c| = \pm 1. \tag{5}$$

One of these possible values of $|a\,b\,c|$ is characteristic of the right-handed trihedrals, and the other of the left-handed trihedrals, as is readily proved by considerations of continuity. Hence:

Theorem 9. *Two rectangular trihedrals are of the same type or of opposite types according as the determinants of the direction cosines of their edges have the same sign or opposite signs.*

For the trihedral whose edges are the positive axes of x, y, z, with the direction cosines 1, 0, 0; 0, 1, 0; 0, 0, 1, the determinant $|a\,b\,c|$ has the value $+\,1$.

Corollary. *A rectangular trihedral whose edges have the oriented directions a, b, c is of the same type as that of the axes if and only if*

$$|a\,b\,c| = 1.$$

Rigid Motions. It is intuitively * evident that there is a unique *real* rigid motion of space which carries the three oriented edges of a given rectangular trihedral into the three corresponding oriented edges of a prescribed rectangular trihedral of the same type. The rigid motion is a *translation,* if corresponding edges of the two trihedrals are parallel and similarly oriented, and a *rotation,* if the two trihedrals have the same vertex.

A given rigid motion may be thought of as determined by any two

* We offer no apologies for the appeal to intuition. We are basing our treatment of metric geometry on the student's previous training in Solid Geometry and this is founded to a large extent, and rightly so, on intuition. An abstract development of metric geometry on a postulational basis is beyond the scope of the present book, and a logical treatment of metric geometry as a subgeometry of projective geometry (Th. 12) before the aspects of metric geometry in extended space are known is as inexpedient as the proverbial cart before the horse.

rectangular trihedrals, the first of which is carried by it into the second. In particular, the first trihedral may always be taken as that of the coordinate axes. Let a, b, c be the oriented directions of the edges, and (d_1, d_2, d_3) the nonhomogeneous coordinates of the vertex, of the second trihedral. The rigid motion is, then, the product of the rotation R carrying the trihedral of the axes into the trihedral whose vertex is at the origin and whose edges have the oriented directions a, b, c, and the translation T which carries the origin into the point (d_1, d_2, d_3). Of course, either R or T may reduce to the identical transformation.

THEOREM 10. *A rigid motion of space is the product of a rotation which leaves the origin fixed and a translation, or either alone.*

The equations of the rotation R and the translation T are

$$R:\quad \begin{aligned} x' &= a_1x + b_1y + c_1z,\\ y' &= a_2x + b_2y + c_2z,\\ z' &= a_3x + b_3y + c_3z, \end{aligned} \qquad\qquad T:\quad \begin{aligned} x' &= x + d_1,\\ y' &= y + d_2,\\ z' &= z + d_3, \end{aligned}$$

as is readily shown by applying the methods by means of which the corresponding transformations of Cartesian coordinates are regularly obtained. Hence, the equations of the general rigid motion are

$$\begin{aligned} x' &= a_1x + b_1y + c_1z + d_1,\\ y' &= a_2x + b_2y + c_2z + d_2,\\ z' &= a_3x + b_3y + c_3z + d_3, \end{aligned} \tag{6}$$

where the (real) coefficients are restricted only by the relations

$$(a|a) = (b|b) = (c|c) = 1, \quad (a|b) = (b|c) = (c|a) = 0, \quad |a\,b\,c| = 1. \tag{7}$$

It is evident from our discussion that there is a one-to-one correspondence between the rigid motions and the rectangular trihedrals of the same type as that of the axes. Hence:

THEOREM 11. *The group of rigid motions depends on six parameters.*

The transformations (6) for which $|a\,b\,c| = -1$ and the remaining relations of (7) remain as before are known as the *reflections of space.* They, too, depend on six parameters, but do not in themselves form a group.

The Transformations of Similarity and the Affine Transformations. The $2\cdot\infty^7$ transformations

$$\begin{aligned} x' &= \rho(a_1x + b_1y + c_1z + d_1),\\ y' &= \rho(a_2x + b_2y + c_2z + d_2),\\ z' &= \rho(a_3x + b_3y + c_3z + d_3), \end{aligned} \tag{8}$$

where

$$(a|a) = (b|b) = (c|c) = 1, \qquad (a|b) = (b|c) = (c|a) = 0, \qquad \rho > 0,$$

are the real *transformations of similarity* of space; in particular, those for which $|a\,b\,c| = 1$ are the *direct*, and those for which $|a\,b\,c| = -1$ are the *indirect*, transformations of similarity.

The ∞^{12} transformations

$$(9) \qquad \begin{aligned} x' &= a_1x + b_1y + c_1z + d_1, \\ y' &= a_2x + b_2y + c_2z + d_2, \\ z' &= a_3x + b_3y + c_3z + d_3, \end{aligned} \qquad |a\,b\,c| \neq 0,$$

are the *affine transformations of space.*

It is evident that the group (6) is contained in the group (8), the group (8) in the group (9), and the group (9) in the group of collineations of extended Cartesian space.

Projective, Affine, and Metric Geometries. These are the geometries associated respectively with the groups of collineations, affine transformations, and rigid motions. Their precise definitions are the same as in the plane (Ch. XI, § 6).

The relationships of the groups (6), (8), (9) to the group of collineations and to one another is readily established.

THEOREM 12. *The affine transformations are the collineations of extended Cartesian space which leave the plane at infinity fixed; the transformations of similarity are the affine transformations which leave the absolute fixed; and the rigid motions and reflections are the transformations of similarity for which distance is an absolute, instead of a relative, invariant.*

We may, then, say that metric geometry is essentially the subgeometry of projective geometry in which a given real plane and a given conic without a real trace lying in this plane are held fast.

EXERCISES

1. Prove Theorem 3.

2. Prove Theorem 8.

3. Establish Laguerre's projective interpretation of angle for two (intersecting) nonisotropic lines in space. Pay special attention to the case in which the two lines lie in an isotropic plane.

4. An angle between two nonisotropic planes is zero if and only if the planes are parallel. Criticize this statement.

5. An imaginary plane is given. How many real points and how many real lines does it contain?

6. An imaginary line is given. How many real points are there on it and how many real planes pass through it?

7. Write a single equation in plane coordinates representing the absolute; see § 4, Ex. 8.

8. Establish relation (5).

9. Count the number of rectangular trihedrals of the same type as that of the axes and hence prove Theorem 11.

10. Prove analytically that the square of the distance between two finite points is an absolute invariant with respect to the group of rigid motions and reflections.

11. Establish Theorem 12.

12. What affine properties come to mind?

13. Show that, if a, b, c are the oriented directions of the edges of a rectangular trihedral of the same type as that of the axes, the relations

$$c_1 = |a_2\, b_3|, \qquad c_2 = |a_3\, b_1|, \qquad c_3 = |a_1\, b_2|$$

hold; also, the relations obtained from them by cyclic advancement of the letters a, b, c.

14. Show that the rotation R is actually a rotation about a fixed line through the origin.

Suggestion. Find the finite fixed points of R.

15. Prove that the rigid motion

$$x' = x\cos\phi - y\sin\phi, \qquad y' = x\sin\phi + y\cos\phi, \qquad z' = z + d_3$$

is a *screw motion* about the z-axis, that is, the product of a rotation about the z-axis and a translation in the direction of the z-axis.

16. Show that *every rigid motion is the product of a rotation about a line and a translation in the direction of the line, or either alone.*

8. Quadric Surfaces. *Point Quadrics.* The totality of points whose coordinates satisfy an equation of the form

$$(1) \qquad \sum_{i,j}^{1-4} a_{ij}x_ix_j = 0, \qquad a_{ij} = a_{ji},$$

where the a's are complex numbers, not all zero, is known as a *point quadric.* The determinant $|a_{ij}|$ is called the *discriminant,* and the rank of the matrix $\|a_{ij}\|$ the *rank,* of the point quadric. When the a's are real or are proportional to a set of real numbers, not all zero, the point quadric is said to be real.

THEOREM 1. *A straight line either intersects a point quadric in two points, distinct or coincident, or lies in the quadric.*

For, if the line is thought of as determined by two points r, s, the points which it has in common with the quadric are the points

$x = \lambda r + \mu s$ for which λ, μ satisfy the equation

$$\lambda^2 \sum a_{ij} r_i r_j + 2\lambda\mu \sum a_{ij} r_i s_j + \mu^2 \sum a_{ij} s_i s_j = 0.$$

THEOREM 2. *A plane either intersects a point quadric in a point conic or is entirely contained in the quadric.*

If r, s, t are three fixed noncollinear points of the plane, an arbitrary point of the plane has coordinates $x = x_1' r + x_2' s + x_3' t$. The result of substituting these coordinates in (1) is a homogeneous quadratic equation in x_1', x_2', x_3'. Since (x_1', x_2', x_3') are projective point coordinates in the plane, this equation represents a point conic or all the points of the plane, according as its coefficients are not, or are, all zero.

If *each* line through a point either intersects a point quadric only in the point, or lies in the quadric, the point is called a *singular point* of the quadric.

THEOREM 3. *The point r is a singular point of the point quadric* (1) *if and only if $\sum a_{ij} r_i y_j \equiv 0$ for all points y.*

The proof is left to the reader. We note that the identity is equivalent to the four equations

$$\sum_{j=1}^{4} a_{ij} r_j = 0, \qquad (i = 1, 2, 3, 4).$$

Therefore, a point quadric has no singular points, one singular point, a range of singular points, or a plane of singular points, according as its rank is four, three, two, or one.

A point quadric which possesses one or more singular points we shall call a *singular point quadric.*

THEOREM 4. *A necessary and sufficient condition that a point quadric be singular is that its discriminant vanish.*

THEOREM 5. *A singular point quadric consists of the points of a cone of lines. The cone is a nondegenerate cone, a pair of distinct planes, or a single doubly counting plane, according as the rank of the quadric is three, two, or one.*

It follows from the definition of a singular point that, if S is a singular point and p is a plane not containing S, the quadric consists of the lines which join S to those points of p which lie on it. Hence, according as p intersects the quadric in a nondegenerate conic, two distinct lines, or a doubly counting line, the quadric is a nondegenerate cone, two distinct planes, or a doubly counting plane.* It is evident geo-

* Not all the points of p belong to the quadric. Why?

metrically that in these three cases the quadric has respectively one singular point, a line of singular points, and a plane of singular points, so that the corresponding values of its rank are respectively three, two, and one.

Nonsingular Point Quadrics. A *tangent line* is defined as a line which meets the quadric in a single (doubly counting) point or lies in the quadric. It is said to be tangent to the quadric at each point which it has in common with the quadric.

THEOREM 6. *The tangent lines to a nonsingular point quadric at a given point form a pencil of lines.*

The plane of the pencil is known as the *tangent plane* at the given point. The theorem is established by showing that the locus of the point y which moves so that the line joining it to the given point r is always tangent to the quadric at r is the plane $\sum a_{ij}r_iy_j = 0$.

THEOREM 7. *The equation of the tangent plane to the nonsingular point quadric* (1) *at the point r is*

$$\sum a_{ij}r_ix_j = 0. \tag{2}$$

Coordinates u of the tangent plane are $u_i = \sum_{j=1}^{4} a_{ij}r_j$, $(i = 1, 2, 3, 4)$. Since $|a_{ij}| \neq 0$, these four equations determine, for given u's not all zero, a unique point r. Hence:

THEOREM 8. *The tangent planes to a nonsingular quadric at two distinct points are distinct.*

The following proposition may be proved in the same way as the corresponding theorem for nondegenerate point conics.

THEOREM 9. *A necessary and sufficient condition that the plane u be tangent to the nonsingular point quadric* (1) *is that*

$$\sum A_{ij}u_iu_j = 0. \tag{3}$$

Here, A_{ij} is the cofactor of a_{ij} in $|a_{ij}|$. Since $a_{ij} = a_{ji}$ and $|a_{ij}| \neq 0$, $A_{ij} = A_{ji}$ and $|A_{ij}| \neq 0$; see Ch. I, § 6, Exs. 2, 3.

Plane Quadrics. The totality of planes whose coordinates satisfy an equation of the form

$$\sum_{i,j}^{1-4} b_{ij}u_iu_j = 0, \qquad b_{ij} = b_{ji}, \tag{4}$$

where the b's are complex numbers, not all zero, is called a *plane quadric*. The determinant $|b_{ij}|$ is known as its discriminant, and the rank of

$\|b_{ij}\|$ as its rank. It is real if the b's are proportional to a set of real numbers, not all zero.

Since in the geometry of planes, a line is a pencil of planes and a point is a sheaf of planes, we agree to mean, by saying that a line or a point lies in, or is contained in, a plane quadric, that all of the planes through it belong to the quadric. The duals of Theorems 1, 2 may then be stated as follows.

THEOREM 10. *A straight line either has two planes in common with a plane quadric or lies in the quadric.*

THEOREM 11. *A point either has the planes of a cone of planes in common with a plane quadric or is contained in the quadric.*

The definitions and discussions of singular planes of a plane quadric and of singular plane quadrics we leave to the reader. He will find:

THEOREM 12. *A necessary and sufficient condition that a plane quadric be singular is that its discriminant vanish. It consists then of the planes of a line conic; the conic is a nondegenerate conic, a pair of distinct points, or a doubly counting point, according as the rank of the quadric is three, two, or one.*

A *tangent line* to a nonsingular plane quadric is a line which has in common with the plane quadric a single (doubly counting) plane or lies in the quadric. It is said to be tangent to the quadric in each plane which it has in common with the quadric.

THEOREM 13. *The tangent lines to a nonsingular plane quadric in a given plane of the quadric form a pencil of lines.*

The vertex of the pencil is known as the *contact point* in the given plane.

THEOREM 14. *The equation of the contact point in the plane r of the nonsingular plane quadric* (4) *is*

$$\sum a_{ij} r_i u_j = 0. \tag{5}$$

THEOREM 15. *A necessary and sufficient condition that the point x be a contact point of the nonsingular plane quadric* (4) *is that*

$$\sum B_{ij} x_i x_j = 0, \qquad B_{ij} = B_{ji}, \quad |B_{ij}| \neq 0. \tag{6}$$

Nonsingular Quadrics. From Theorems 9 and 15 we conclude

THEOREM 16. *The nonsingular point quadrics and the nonsingular plane quadrics correspond in pairs. If Q is the point quadric and Q' the plane quadric of a pair, the tangent planes of Q are the planes of Q' and the contact points of Q' are the points of Q.*

Thus, the configurations obtained by adjoining to the nonsingular point quadrics their tangent planes and those obtained by adjoining to the nonsingular plane quadrics their contact points are identical. We shall call them *nonsingular quadrics.*

If (1) is the equation satisfied by the coordinates of the points of a nonsingular quadric, then (3) is the equation satisfied by the coordinates of the planes of the quadric. Or, if (4) is the equation of the quadric in plane coordinates, (6) is its equation in point coordinates.

The Rulings of a Nonsingular Quadric. From the definition of a tangent line to a point (plane) quadric, it is evident that, if two distinct points (planes) of a tangent line to a nonsingular quadric belong to the quadric, all the points (planes) of the line belong to the quadric. But, if two distinct points of a tangent line lie on a quadric, the tangent planes at these points are two distinct planes (Th. 8) passing through the line and belonging to the quadric, and vice versa. Hence:

THEOREM 17. *If all the points of a line lie on a nonsingular quadric, all the planes through the line belong to the quadric, and conversely.*

A line all of whose points or all of whose planes belong to the quadric is known as a *ruling* of the quadric.

THEOREM 18. *Two distinct rulings lie in each plane, and two distinct rulings pass through each point, of a nonsingular quadric.*

It suffices to demonstrate the second of the two contentions. Let r be a point of the quadric and L a line not through r lying in the tangent plane at r. The rulings through r are the lines joining r to the points in which L intersects the quadric. These are the points $x = \lambda s + \mu t$ for which

$$\lambda^2 \sum a_{ij} s_i s_j + 2\lambda\mu \sum a_{ij} s_i t_j + \mu^2 \sum a_{ij} t_i t_j = 0,$$

where s and t are two distinct points on L.

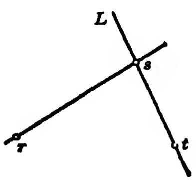


FIG. 3

We may assume that s is a point of intersection of L with the quadric: $\sum a_{ij} s_i s_j = 0$. Then the expression $\sum a_{ij} s_i t_j$ is not zero; for, if it were zero, the tangent plane at s would contain the point t and hence coincide with the tangent plane at r, since in any case it contains the ruling joining r to s (Fig. 3).

Inasmuch as $\sum a_{ij} s_i s_j = 0$ and $\sum a_{ij} s_i t_j \neq 0$, L intersects the quadric in two distinct points and there are two distinct rulings through r. Thus the theorem is proved.

Denote by L_0, M_0 the two rulings which pass through a given point P_0 and hence lie in the tangent plane p_0 at P_0, and consider the rulings M which meet L_0 and the rulings L which meet M_0 (Fig. 4). These two families of rulings exhaust all the rulings; for, an arbitrary ruling, other than L_0 and M_0, must intersect p_0 in a point of the quadric, that is, in a point on L_0 or M_0.

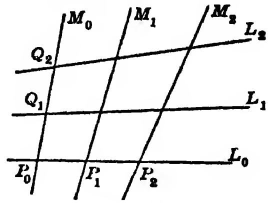


FIG. 4

The two families of rulings are mutually exclusive, since, if they had a line in common, there would be three rulings in p_0 or three rulings through P_0.

Two rulings of the same family are skew lines. For, if M_1, M_2, for example, intersected, the tangent planes at P_1, P_2 would be identical.

Two rulings of opposite families always intersect. The rulings M_1, L_1 lie respectively in the tangent planes at P_1, Q_1 and intersect on the line common to these planes, since otherwise this line would contain three distinct points of the quadric.

It follows that each family consists of all the rulings which intersect a ruling of the other family. Thus, our definition of the two families is independent of the particular rulings L_0, M_0 originally chosen.

The families of rulings are known as the *reguli* of the quadric, and each regulus is said to be *conjugate* to the other.

THEOREM 19. *There is one ruling of each regulus in each plane, and one ruling of each regulus through each point, of the quadric. Two rulings are skew or intersecting lines according as they do, or do not, belong to the same regulus.*

The points and planes of the rulings of *either* regulus exhaust just once the points and planes of the quadric.

EXERCISES

1. Show that a plane always intersects a nonsingular point quadric in a point conic and that the conic is nondegenerate or consists of two rulings according as the plane is not, or is, a tangent plane.

2. State and prove the dual of the previous exercise.

3. Prove the following theorems:
(*a*) Theorem 3; (*b*) Theorem 12; (*c*) Theorem 13.

4. A *skew quadrilateral* may be defined as the configuration obtained by suppressing two opposite edges of a tetrahedron. Show that there exist ∞^4 skew quadrilaterals which lie on a given nonsingular quadric, that is, whose edges, vertices, and faces are respectively rulings, points, and planes of the quadric.

5. Prove that each diagonal of a skew quadrilateral on a nonsingular quadric contains the contact points of the faces through the other diagonal and lies in the tangent planes at the vertices on the other diagonal.

6. Show that there is a unique skew quadrilateral on a given nonsingular quadric which has a given line not tangent to the quadric as a diagonal.

7. Prove that two points, one on each diagonal of a skew quadrilateral on a nonsingular quadric, are separated harmonically by the points in which their line meets the quadric, provided that neither point is a vertex of the quadrilateral.

8. Two point conics lying in different planes are tangent to the line of intersection of the two planes at the same point. How many point quadrics are there which contain both conics?

9. The previous exercise, when the two conics meet the line of intersection of the two planes in the same points.

10. The bilinear forms

$$\Sigma a_{ij}x_iy_j, \qquad a_{ij} = a_{ji}, \qquad \Sigma b_{ij}x_iy_j, \qquad b_{ij} = b_{ji},$$

are known as the *polar* forms of the quadratic forms

$$\Sigma a_{ij}x_ix_j, \qquad a_{ij} = a_{ji}, \qquad \Sigma b_{ij}x_ix_j, \qquad b_{ij} = b_{ji}.$$

Show that if the quadratic forms, set equal to zero, represent the same nonsingular quadric, the polar forms, equated to zero, represent the same involutory correlation.

9. The Polar System of a Quadric. In Ch. XIV we developed, on its own merits, the relation of pole and polar with respect to a nondegenerate conic, only to find afterwards that we were dealing simply with an involutory correlation. Let us here reverse the process and define, as the polar system of a given nonsingular quadric, the involutory correlation which is associated with the quadric, in the sense of § 8, Ex. 10.

General Properties. In an involutory correlation, the points and planes of space, and also the lines of space, correspond in pairs. For, any correlation carries a point into a plane, a plane into a point, and a line into a line, and an involutory correlation, since it is its own inverse, interchanges the elements of space in pairs.

Two elements paired by the polar system of the given quadric are known as *polar elements* with respect to the quadric. A point and a plane which correspond are referred to as *pole* and *polar*, and two corresponding lines are called *polar lines*.

THEOREM 1. *If the first of two elements is in united position with the polar element of the second, the second element is in united position with the polar element of the first.*

Let the two elements be E_1, E_2 and their polar elements, E_1', E_2'. We are to show that, if E_1 and E_2' have united position, E_2 and E_1' also have united position. We remark, to begin with, that every correlation preserves united position. Hence, since E_1 and E_2' have united position, E_1' and the polar element of E_2' have united position. But the polar element of E_2' is E_2, since our correlation is involutory. Therefore, E_1' and E_2 have united position, and the theorem is proved.

DEFINITION. *Two elements each of which has united position with the polar element of the other are said to be conjugate elements with respect to the quadric.*

We paraphrase the definition in two important cases.

Two points each of which lies in the polar plane of the other are *conjugate points.*

Two planes each of which contains the pole of the other are *conjugate planes.*

As an immediate consequence of these definitions, we have

THEOREM 2 *a*. *The points conjugate to a given point are the points of the polar plane of the given point.*

THEOREM 2 *b*. *The planes conjugate to a given plane are the planes through the pole of the given plane.*

The polar of a line is the line of intersection of the polar planes of two points on the line, or the line joining the poles of two planes through the line. Hence:

THEOREM 3 *a*. *The points conjugate to two given points and hence to all the points of their line are the points of the polar line.*

THEOREM 3 *b*. *The planes conjugate to two given planes and hence to all the planes through their line are the planes through the polar line.*

Specific Description. The foregoing general properties of a polar system may well be called the *involutory properties* of the system, for they depend only on the fact that the polar system is its own inverse. We shall now write the actual equations of the system and deduce from them its geometric relations to the quadric.

If the equations in associated point and plane coordinates of the given nonsingular quadric are respectively*

$$(1) \qquad \sum a_{ij}x_ix_j = 0, \quad a_{ij} = a_{ji}, \qquad \sum b_{ij}u_iu_j = 0, \quad b_{ij} = b_{ji},$$

the bilinear equations in point and plane coordinates of the polar

* According as the first or second of the equations (1) is thought of as the one originally given, $b_{ij} = \lambda A_{ij}$, $\lambda \neq 0$ or $a_{ij} = \mu B_{ij}$, $\mu \neq 0$.

system are (§ 8, Ex. 10)

(2) $\sum a_{ij}x_iy_j = 0,$ $\sum b_{ij}u_iv_j = 0.$

THEOREM 4 *a*. *The polar of the point r with respect to* (1) *is the plane*

(3 *a*) $\sum a_{ij}r_ix_j = 0.$

THEOREM 4 *b*. *The pole of the plane r with respect to* (1) *is the point*

(3 *b*) $\sum b_{ij}r_iu_j = 0.$

It follows that a point on the quadric and the tangent plane at the point are pole and polar.

THEOREM 5 *a*. *The two points r, s are conjugate with respect to* (1) *if and only if*

$$\sum a_{ij}r_is_j = 0.$$

THEOREM 5 *b*. *The two planes r, s are conjugate with respect to* (1) *if and only if*

$$\sum b_{ij}r_is_j = 0.$$

In particular, a point or a plane is self-conjugate when and only when it belongs to the quadric.

THEOREM 6 *a*. *Two points, neither of which is on the quadric, are conjugate if and only if they separate harmonically the points which their line has in common with the quadric. Two points, the first of which is on the quadric, are conjugate when and only when the second lies in the tangent plane at the first.*

THEOREM 6 *b*. *Two planes, neither of which belongs to the quadric, are conjugate if and only if they separate harmonically the planes which their line has in common with the quadric. Two planes, the first of which belongs to the quadric, are conjugate when and only when the second contains the contact point of the first.*

Consider, now, a point P and a plane p which are pole and polar and do not belong to the quadric. The plane p has in common with the quadric a nondegenerate conic c and the point P has in common with the quadric a nondegenerate cone C(§ 8, Exs. 1, 2). A tangent plane to the quadric at a point of the conic c is conjugate to p (Th. 6 *b*), and therefore goes through P and is a plane of the cone C. Dually, the point of contact with the quadric of a plane of the cone C is conjugate to P (Th. 6 *a*), and so lies in p and is a point of the conic c. Hence, the points of contact with the quadric of the planes of C are the points of c and the tangent planes to the quadric at the points of c are the planes of C.

We shall call c the conic of contact of the cone C, and shall describe C as the cone enveloping the quadric along the conic c.

THEOREM 7 *a*. *The polar of a point not on the quadric is the plane of the conic of contact of the cone which the quadric subtends at the point. The polar of a point on the quadric is the tangent plane at the point.*

THEOREM 7 *b*. *The pole of a plane not belonging to the quadric is the vertex of the cone enveloping the quadric along the conic in which the plane intersects the quadric. The pole of a plane belonging to the quadric is the contact point of the plane.*

We leave it to the reader to establish the following facts concerning polar lines.

THEOREM 8. *A ruling of the quadric is self-polar, two tangent lines (neither a ruling) which have the same point of contact and separate harmonically the rulings through the point are polars of one another, and two nontangent lines which are the diagonals of a skew quadrilateral on the quadric are mutually polar.*

According to the definition of conjugate elements, two lines each of which intersects the polar of the other are *conjugate lines*. In particular, a line is self-conjugate when it intersects its own polar and hence, by Th. 8, when it is a tangent line.

THEOREM 9. *The self-conjugate points, the self-conjugate planes, and the self-conjugate lines are the points, the planes, and the tangent lines of the quadric.*

It is worth while noting that two polar lines constitute a special case of two conjugate lines, and that a self-polar line, that is, a ruling, is a special case of a self-conjugate line.

Self-Conjugate Tetrahedra. A tetrahedron is self-conjugate if each vertex and the opposite face are pole and polar.

THEOREM 10. *There are ∞^6 tetrahedra which are self-conjugate with respect to the given quadric.*

The proof is similar to that for the corresponding theorem for conics.

THEOREM 11. *A necessary and sufficient condition that a tetrahedron be self-conjugate is that each two vertices, or each two faces, or each two edges, be conjugate, or that each two opposite edges be polar lines.*

We leave the proof to the reader.

Projective Classification of Nonsingular Quadrics. By introducing new associated point and plane coordinates referred to a self-conjugate tetrahedron as tetrahedron of reference, and to a properly chosen point and plane as unit point and unit plane, equations (1) of our quadric may be reduced simultaneously to the forms

$$(4) \qquad x_1^2 + x_2^2 + x_3^2 + x_4^2 = 0, \qquad u_1^2 + u_2^2 + u_3^2 + u_4^2 = 0.$$

THEOREM 12. *Each two nonsingular quadrics are equivalent with respect to the group of complex collineations.*

Thus, in *complex* projective geometry, all nonsingular quadrics are of the same type. On the other hand, in *real* projective geometry there are three types of nonsingular quadrics, corresponding to the three canonical point equations

$$(5) \qquad \begin{aligned} x_1^2 + x_2^2 + x_3^2 + x_4^2 &= 0, \\ x_1^2 + x_2^2 + x_3^2 - x_4^2 &= 0, \\ x_1^2 + x_2^2 - x_3^2 - x_4^2 &= 0. \end{aligned}$$

A quadric of the first type is without a real trace and hence has imaginary rulings, one of the second type has a real trace but imaginary rulings, and one of the third type has both a real trace and real rulings.

EXERCISES

1. Show that the cross ratios of four points on a ruling are equal to the cross ratios of the tangent planes at the four points.

2. Prove Theorem 8. Show that, if a ruling intersects one of two polar lines, it intersects the other.

3. Two lines are polar if and only if there exist at least two points on the one whose polar planes contain the other. Show that two lines are conjugate if and only if there exists at least one point on the one whose polar plane contains the other. State the corresponding properties for a self-conjugate line and a self-polar line.

4. How many lines are there conjugate to a given line? Show that the lines of a pencil of lines whose vertex and plane do not belong to the quadric are conjugate in pairs, and that the pairs form an involution.

5. Show that, if a point and a plane are conjugate, they are pole and polar.

6. The lines conjugate to a given point are the lines in its polar plane, the lines conjugate to a given plane are the lines through its pole, and the points and planes conjugate to a given line are the points and planes of its polar line. Establish these propositions.

7. Show that the planes and lines through a point are conjugate in pairs, provided the point is not on the quadric. State the dual.

8. Prove that the polar system with respect to the quadric establishes in a plane the polar system with respect to the conic in which the plane intersects the quadric. State the dual.

Note that, if a point and a line are conjugate with respect to a nondegenerate conic, they are pole and polar; see Ex. 5.

9. Establish directly the theorem: The elements conjugate to a given element are the elements in united position with its polar element.

10. Prove Theorem 11.

11. Show that a triangle which is self-conjugate with respect to the conic in which its plane meets the quadric determines with the pole of the plane a self-conjugate tetrahedron, provided the plane does not belong to the quadric.

12. Verify the classification of real quadrics with respect to real collineations.

13. A nonsingular quadric is referred to a tetrahedron consisting of a skew quadrilateral on the quadric and its diagonals, as tetrahedron of reference, and to a point on the quadric as unit point. Show that, if the diagonals of the quadrilateral are the edges A_2A_3, A_1A_4 of the tetrahedron, the equation in point coordinates of the quadric is

$$x_2x_3 = x_1x_4.$$

What is its equation in the associated plane coordinates?

10. Affine and Metric Properties of Quadrics. The *affine properties* of a nonsingular quadric bear on the relationship of the quadric to the fixed plane, the plane at infinity, of affine geometry.

According as the quadric is, or is not, tangent to the plane at infinity, it is known as a *paraboloid* or a *central quadric*.

Characteristic of a *center*, that is, a point of symmetry, of a quadric is that it be a finite point conjugate to every point at infinity. Hence:

THEOREM 1. *A central quadric has one and only one center, the pole of the plane at infinity. A paraboloid has no center.*

The finite planes which are tangent to a quadric at its points at infinity form a cone of planes, which is called *the asymptotic cone.* In the case of a central quadric, the asymptotic cone is the nondegenerate cone which the quadric subtends at its center. The asymptotic cone of a paraboloid is degenerate.

The finite lines and planes which pass through the pole of the plane at infinity are known respectively as the *diameters* and *diametral planes* of the quadric. The discussion of their properties we leave to the reader (Exs. 3–5).

Affine Classification. In affine geometry, we are restricted to tetrahedra of reference which have the plane at infinity as their common fourth face, $x_4 = 0$. The corresponding coordinates are called *affine*

coordinates. In particular, affine point coordinates are simply oblique Cartesian coordinates; see Ch. XI, § 4.

A permissible tetrahedron of reference which is self-conjugate with respect to a given central quadric is determined by the center A_4 of the quadric and a triangle $A_1A_2A_3$ at infinity which is self-conjugate with respect to the conic at infinity on the quadric. The equation in point coordinates of the quadric, referred to this tetrahedron, is reducible to the form

$$(1) \qquad x_1^2 + x_2^2 + x_3^2 + x_4^2 = 0$$

by a suitable choice of the unit point.

There is no permissible tetrahedron of reference which is self-conjugate with respect to a paraboloid. However, a tetrahedron well suited to our purpose is obtained by taking as the vertices A_1, A_2 two conjugate points at infinity (neither self-conjugate), as A_3 the contact point of the plane at infinity, and as A_4 the contact point of the finite tangent plane through the line A_1A_2. The vertices A_3, A_4 of this tetrahedron are on the paraboloid and each two vertices, except A_3, A_4, are conjugate. Hence, in our equation of the paraboloid in point coordinates, all the coefficients are zero except a_{11}, a_{22} and a_{34}, a_{43}. By a proper choice of the unit point, it is possible to reduce the equation to the form

$$(2) \qquad x_1^2 + x_2^2 = 2\, x_3x_4.$$

THEOREM 2. *In complex affine geometry there is just one type of central quadric and one type of paraboloid.*

The classification of *real* nonsingular quadrics with respect to *real* affine transformations yields four types of central quadrics and two types of paraboloids. The central quadrics fall immediately into the *ellipsoids* and the *hyperboloids*, according as their conics at infinity have not, or have, real traces, and it turns out that there are two types of ellipsoids and two types of hyperboloids. Canonical point equations, written in nonhomogeneous affine coordinates, are

$$(3) \qquad \begin{aligned} &x^2 + y^2 + z^2 + 1 = 0\text{: ellipsoid without a real trace;}\\ &x^2 + y^2 + z^2 - 1 = 0\text{: ellipsoid with a real trace;}\\ &x^2 + y^2 - z^2 + 1 = 0\text{: hyperboloid without real rulings;}\\ &x^2 + y^2 - z^2 - 1 = 0\text{: hyperboloid with real rulings.}\end{aligned}$$

A paraboloid is either *elliptic* or *hyperbolic*, according as its rulings at infinity are conjugate-imaginary or real. There is only one type of

elliptic paraboloid and one type of hyperbolic paraboloid:

$$\text{(4)}\qquad \begin{aligned} x^2 + y^2 &= 2z\text{: paraboloid without real rulings;} \\ x^2 - y^2 &= 2z\text{: paraboloid with real rulings.} \end{aligned}$$

Metric Properties and Classification. We have to do here with the relationship of the quadric to the absolute conic. We shall restrict ourselves to real quadrics and be content with the discussion of the general case of a real central quadric whose conic at infinity intersects the absolute in four distinct points. In this case there is a *unique* triangle $A_1A_2A_3$ in the plane at infinity which is self-conjugate with respect to both the absolute and the conic at infinity on the quadric. This triangle and the center A_4 of the quadric determine a real tetrahedron $A_1A_2A_3A_4$ which is self-conjugate with respect to the quadric. Furthermore, the finite edges (faces) of the tetrahedron are mutually perpendicular (§ 7, Th. 8). The tetrahedron may, then, be employed as the tetrahedron of reference for a system of rectangular Cartesian point coordinates. The equation of the quadric in these coordinates is of the form

$$\text{(5)}\qquad ax^2 + by^2 + cz^2 = 1,$$

where a, b, c are real numbers, no two equal and no one zero.

The finite edges and the finite faces of the tetrahedron $A_1A_2A_3A_4$, since they are mutually perpendicular and conjugate in pairs, are respectively *axes*, that is, lines of symmetry, and *principal planes*, that is, planes of symmetry, of the quadric. Moreover, because of the uniqueness of the triangle $A_1A_2A_3$, they are the only axes and principal planes.

THEOREM 3. *A real central quadric has, in general, three axes and three principal planes. They are the edges and the faces of a rectangular trihedral whose vertex is the center of the quadric.*

We conclude with a discussion of the circles on the general quadric. A section of the quadric by a plane is a circle only when its points at infinity are points of the absolute. But these points are in any case points of the conic at infinity on the quadric. Hence a (finite) plane cuts the quadric in a circle if and only if its line at infinity is a common chord of the absolute conic and the conic at infinity on the quadric. Since, by hypothesis, these two conics have four distinct points of intersection which are conjugate-imaginary in pairs, they have six distinct common chords, two of which are real. Thus:

THEOREM 4. *There exist in general six distinct pencils of parallel planes which intersect a real quadric in circles. The circular sections by the planes of two of the pencils are real.*

The relationship of the six pencils of planes to the three principal planes follows readily from the relationship of the common chords of the two conics to their common self-conjugate triangle.

EXERCISES

1. Show that every section of a central quadric by a plane not parallel to, or coincident with, a plane of the asymptotic cone is a central conic. What is the section by a plane parallel to a plane of the asymptotic cone? The section by a plane of the asymptotic cone?

2. Discuss the plane sections of a paraboloid.

3. Prove that there is a unique diametral plane of a central quadric which is conjugate to a given diameter, and that the line at infinity in the plane and the point at infinity on the diameter are polar and pole with respect to the conic at infinity on the quadric. Discuss conjugate diametral planes and conjugate diameters.

4. The locus of the mid-points of the chords of a central quadric which are parallel to a given diameter is the conjugate diametral plane, and the locus of the centers of the sections of the quadric by planes parallel to a given diametral plane is the conjugate diameter. Show that these theorems are true in general, and determine the exceptions.

5. State and prove the theorems for a paraboloid which are analogous to those of Ex. 4 for a central quadric.

6. A *cylinder* is defined as a cone whose vertex is a point at infinity. When will the cone which envelopes a quadric along a given conic be a cylinder?

7. Verify the fact that in real affine geometry there are six types of real nonsingular quadrics.

8. Prove that a line is an axis of a nonsingular quadric if and only if it is a diameter and is perpendicular to a plane conjugate to it. State and prove similar conditions that a plane be a principal plane.

9. Give a complete classification of real central quadrics with respect to real rigid motions. Determine in each case the number of axes and principal planes and the number of circles on the quadric, including circles in isotropic planes; see Exs. 13, 14.

10. The preceding exercise for paraboloids.

11. In real affine geometry, there are three general types of cylinders, the elliptic, parabolic, and hyperbolic. Are there not, then, three general types of cones?

12. By the methods of the text find the vertex, axis, and principal planes of the hyperbolic paraboloid

$$x^2 - y^2 - z^2 + 2\,yz + 2x - y + 3\,z + 4 = 0.$$

13. A circle in an isotropic plane is defined as a point conic in the plane which is tangent to the line at infinity at *the* circular point at infinity. Show that the section of a sphere by an isotropic plane is a circle. Prove that there are ∞^1 spheres through a given circle in an isotropic plane (see § 8, Ex. 8), and show that the loci of their centers is a straight line. How is this line related to the isotropic plane?

14. Locate the center of a nondegenerate circle in an isotropic plane, if the center is defined as the point of intersection of the plane and the line joining the pole of the plane, with respect to a nondegenerate sphere containing the circle, with the center of the sphere.

11. Projective Generation of Quadrics. Projective Geometry on a Quadric. We return to the projective geometry of a complex nonsingular quadric and take, as the equation in point coordinates of the quadric,

$$x_2x_3 = x_1x_4. \tag{1}$$

This is the form assumed by the point equation when the opposite edges A_1A_2, A_3A_4 and A_1A_3, A_2A_4 of the tetrahedron of reference are arbitrarily chosen rulings of the quadric, two from each regulus, and the unit point lies on the quadric (§ 9, Ex. 13).

THEOREM 1. *The equations of the reguli of the quadric* (1) *are*

$$R_1:\ \begin{matrix} x_2 = u\,x_4, \\ u\,x_3 = x_1, \end{matrix} \qquad\qquad R_2:\ \begin{matrix} x_3 = v\,x_4, \\ v\,x_2 = x_1, \end{matrix}$$

where u and v are parameters.

The proof is left to the reader. In particular, the rulings A_1A_2, A_3A_4 are the lines $u = \infty$, $u = 0$ of the regulus R_1, and A_1A_3, A_2A_4 are the lines $v = \infty$, $v = 0$ of the regulus R_2.

The two planes

$$x_2 = u\,x_4, \qquad\qquad x_1 = u\,x_3, \tag{2}$$

which intersect in the general line L of R_1, pass respectively through the rulings A_1A_3, A_2A_4 of R_2. As L traces R_1, the planes generate the two pencils of planes with A_1A_3, A_2A_4 as axes. Since corresponding planes (2) have the same projective coordinate u in their respective pencils, the two pencils are projective. Recalling that A_1A_3, A_2A_4 were arbitrarily chosen rulings of R_2, we conclude

THEOREM 2. *The lines of a regulus determine with two lines of the conjugate regulus projective pencils of planes.*

THEOREM 3. *Corresponding planes of two projective pencils of planes with skew axes intersect in the lines of a regulus. The axes of the pencils belong to the conjugate regulus.*

The latter theorem is true inasmuch as coordinates in space and in the pencils may be so chosen that the projective correspondence between the two pencils is represented by equations (2).

The pencil of planes p through A_1A_3 cuts A_2A_4 in a range of points Q, and the pencil of planes q through A_2A_4 cuts A_1A_3 in a range of points P (Fig. 5). Since the pencils p and q are projective, the ranges Q and P which are respectively perspective to them are projective. But the lines PQ are the lines L of R_1. Thus:

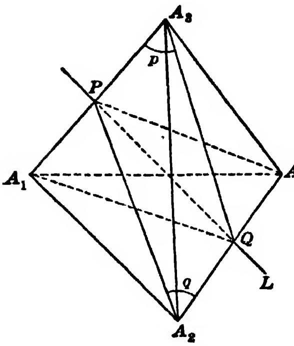


Fig. 5

Theorem 4. *The lines of a regulus cut two lines of the conjugate regulus in projective ranges of points.*

We give also the analytic proof. The ranges of points P, Q in which the lines L of R_1 meet A_1A_3, A_2A_4 are

$$(3) \qquad (u, 0, 1, 0), \qquad (0, u, 0, 1).$$

Since for corresponding points u has the same value, the ranges are projective.

Theorem 5. *The lines joining corresponding points of two projective ranges of points lying on skew lines are the lines of a regulus. The lines of the ranges belong to the conjugate regulus.*

The method of proof is the same as that for Theorem 3.

Theorem 6. *There is one and only one regulus which contains three given mutually skew lines.*

Let the three given lines be L_1, L_2, L_3, and let M_1, M_2 be two arbitrarily chosen lines which intersect L_1, L_2, L_3, in the points A_1, B_1, C_1 and A_2, B_2, C_2. Establish the projective correspondence between the ranges of points on M_1, M_2 determined by the three pairs of corresponding points $A_1 \longleftrightarrow A_2$, $B_1 \longleftrightarrow B_2$, $C_1 \longleftrightarrow C_2$. The lines joining corresponding points of the two ranges are, by Th. 5, the lines of a regulus, R_1, which contains L_1, L_2, L_3.

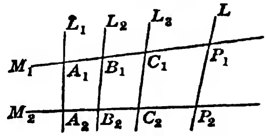


Fig. 6

Suppose that R_1' is a second regulus to which L_1, L_2, L_3 belong. Then M_1, M_2 belong to the regulus conjugate to R_1', since each has

three distinct points in common with the quadric determined by R_1'. Hence, by Th. 4, the lines of R_1' cut M_1, M_2 in two projective ranges of points for which $A_1 \longleftrightarrow A_2$, $B_1 \longleftrightarrow B_2$, $C_1 \longleftrightarrow C_2$ are pairs of corresponding points. Thus, R_1' answers to the same description as R_1 and coincides with R_1. The theorem is, then, established.

Since the lines M_1, M_2 were arbitrary lines intersecting L_1, L_2, L_3, every line intersecting L_1, L_2, L_3 belongs to the regulus R_2 conjugate to R_1. Conversely, every line of R_2 intersects L_1, L_2, L_3. Hence:

THEOREM 7. *The lines which intersect three given mutually skew lines are the lines of a regulus. The three given lines belong to the conjugate regulus.*

As a consequence of either Theorem 6 or 7, we have

THEOREM 8. *There is a unique nonsingular quadric having three given mutually skew lines as rulings.*

THEOREM 9. *There is a unique nonsingular quadric containing a given skew quadrilateral and a point not lying in a face of the tetrahedron determined by the quadrilateral.*

Theorem 9 may be established directly, or by means of Theorem 8.

Projective Geometry in a Regulus. A regulus is a one-dimensional fundamental form of the second order and, like the previous forms of this description,* possesses the usual one-dimensional projective geometry.

The cross ratio (L_1L_2, L_3L_4) of four distinct lines of a regulus R_1 is defined as the cross ratio (P_1P_2, P_3P_4) of the four points in which the lines intersect a ruling M of the conjugate regulus R_2, or as the cross ratio (p_1p_2, p_3p_4) of the four planes which the lines determine with M. Since p_1, p_2, p_3, p_4 are the tangent planes at P_1, P_2, P_3, P_4 to the quadric determined by R_1, the cross ratios (P_1P_2, P_3P_4) and (p_1p_2, p_3p_4) are actually equal (§ 9, Ex. 1). That they are independent of the particular ruling, M, chosen from R_2 is clear from Theorems 2, 4.

A cross ratio of four lines of our regulus R_1, expressed analytically in terms of the values of u for the four lines, evidently has its customary form. Hence, the parameter u, which we have already recognized as a projective coordinate in the pencils of planes (2) and the ranges of

* These are: the points of a nondegenerate conic, the lines of a nondegenerate conic (Ch. XV, § 9) and their space duals, the planes of a nondegenerate cone and the lines of a nondegenerate cone; the regulus completes the list.

points (3), is also a projective coordinate in the regulus R_1. Similarly, v is a projective coordinate in the regulus R_2.

The corresponding homogeneous coordinates (u_1, u_2) and (v_1, v_2) are obtained in the usual way by setting

$$(4) \qquad \frac{u_1}{u_2} = u, \qquad \frac{v_1}{v_2} = v.$$

The theory of projective transformations of a one-dimensional form into itself, or into a second one-dimensional form, applies without change to reguli. Thus the pairs of equations

$$\begin{matrix} \rho u_1' = a_1 u_1 + a_2 u_2, \\ \rho u_2' = b_1 u_1 + b_2 u_2, \end{matrix} \quad |a\, b| \neq 0, \qquad \begin{matrix} \rho v_1' = c_1 u_1 + c_2 u_2, \\ \rho v_2' = d_1 u_1 + d_2 u_2, \end{matrix} \quad |c\, d| \neq 0,$$

represent, respectively, the general projective transformations of the regulus R_1 into itself and into the regulus R_2. The first pair of equations may also be interpreted as a change to new projective coordinates in R_1.

Projective Geometry on a Quadric. Through a given point x of the quadric (1) there passes a unique ruling (u_1, u_2) and a unique ruling (v_1, v_2). Conversely, two given rulings (u_1, u_2), (v_1, v_2) determine a unique point x of the quadric. Thus there is a perfect one-to-one correspondence between the points x of the quadric and the pairs of rulings (u_1, u_2), (v_1, v_2).

The equations of the correspondence may be obtained by solving the equations of Theorem 1, after introducing the homogeneous parameters (4), for $x_1 : x_2 : x_3 : x_4$ or for $u_1 : u_2$ and $v_1 : v_2$. They turn out to be

$$(5) \quad \rho x_1 = u_1 v_1, \qquad \rho x_2 = u_1 v_2, \qquad \rho x_3 = u_2 v_1, \qquad \rho x_4 = u_2 v_2,$$

$$(6) \quad u_1 : u_2 = x_1 : x_3 = x_2 : x_4, \qquad v_1 : v_2 = x_1 : x_2 = x_3 : x_4.$$

Since to a given point of the quadric there correspond unique ratios $u_1 : u_2$, $v_1 : v_2$, neither 0, 0, and conversely, $(u_1, u_2; v_1, v_2)$ constitute *mixed* homogeneous coördinates for the points of the quadric. We shall call them homogeneous *projective* point coordinates for the quadric.*

* In the corresponding system of nonhomogeneous coordinates (u, v), the points of the rulings $A_1A_2 : u = \infty$ and $A_1A_3 : v = \infty$ which pass through A_1 are the "infinite points," that is, the points which, strictly speaking, have no coordinates. In light of this fact, let the reader compare the mixed homogeneous projective coordinates for the points of the quadric (1) with the mixed homogeneous isotropic coordinates of Ch. XVIII, § 18 for the points of the complex inversive plane.

Equations (5) may be thought of as constituting the *parametric representation* of the quadric (1) in terms of the coordinates $(u_1, u_2; v_1, v_2)$ on the quadric, or as representing the change from these coordinates to the space coordinates (x_1, x_2, x_3, x_4).

THEOREM 10. *The equation in the coordinates on the quadric of the conic in which the plane* $(a|x) = 0$ *intersects the quadric is*

$$a_1u_1v_1 + a_2u_1v_2 + a_3u_2v_1 + a_4u_2v_2 = 0. \tag{7}$$

Conversely, every equation bilinear in u_1, u_2 *and* v_1, v_2, *not all of whose coefficients are zero, represents a conic on the quadric.*

The result of carrying out simultaneously a projective transformation of the rulings of R_1 into the rulings of R_1 and a projective transformation of the rulings of R_2 into the rulings of R_2 is a transformation

$$\text{(8 a)}\qquad \begin{aligned} \rho u_1' &= a_1u_1 + a_2u_2, & \sigma v_1' &= c_1v_1 + c_2v_2, & |a\,b| &\neq 0,\\ \rho u_2' &= b_1u_1 + b_2u_2, & \sigma v_2' &= d_1v_1 + d_2v_2, & |c\,d| &\neq 0, \end{aligned}$$

of the points of the quadric into themselves. Similarly, the transformation

$$\text{(8 b)}\qquad \begin{aligned} \rho v_1' &= a_1u_1 + a_2u_2, & \sigma u_1' &= c_1v_1 + c_2v_2, & |a\,b| &\neq 0,\\ \rho v_2' &= b_1u_1 + b_2u_2, & \sigma u_2' &= d_1v_1 + d_2v_2, & |c\,d| &\neq 0, \end{aligned}$$

which is the result of simultaneous projective transformations of the rulings of R_1 into those of R_2 and the rulings of R_2 into those of R_1, is also a transformation of the points of the quadric into themselves.

THEOREM 11. *A collineation of space which leaves the quadric in place establishes on the quadric a transformation* (8 a) *or* (8 b). *Conversely, there exists a unique collineation of space which carries the quadric into itself and establishes on it a given transformation* (8 a) *or* (8 b).

A collineation which carries the quadric into itself carries skew lines on the quadric into skew lines on the quadric and hence transforms each regulus of the quadric into itself or into the opposite regulus. Since these transformations of the reguli are necessarily projective, the transformation of the points of the quadric is of the form (8 a) or (8 b).

Conversely, let T be a transformation on the quadric which we assume, to be explicit, to be of type (8 a), and let T_1 and T_2 be respectively the component projective transformations of the reguli R_1 and R_2. Think of T_1 as determined by three distinct rulings L_1, L_2, L_3 of R_1 and their transforms L_1', L_2', L_3', and of T_2 as similarly determined by $M_1 \longleftrightarrow M_1'$, $M_2 \longleftrightarrow M_2'$, $M_3 \longleftrightarrow M_3'$. Form the skew quadrilateral determined by L_1, L_2, M_1, M_2 and the corresponding skew quadrilateral

determined by L_1', L_2', M_1', M_2'; and mark the point P of intersection of L_3, M_3 and the point P' of intersection of L_3', M_3'. There exists a unique collineation of space which carries P and the four vertices of the first quadrilateral respectively into P' and the corresponding four vertices of the second quadrilateral. By Th. 9, this collineation carries the quadric into itself. It obviously carries L_1, L_2, M_1, M_2, into L_1', L_2', M_1', M_2', and, since L_3, M_3 are the rulings of the quadric through P and L_3', M_3' those through P', it carries L_3, M_3 into L_3', M_3'. The projective transformations of the reguli R_1, R_2 established by it are, then, T_1 and T_2. Therefore the transformation of the points of the quadric which is established by it is the given transformation T.

The theorem justifies us in defining the transformations (8 *a*) and (8 *b*) as *the* projective transformations of the quadric into itself.

THEOREM 12. *The group of projective transformations of a quadric into itself consists of $2 \cdot \infty^6$ transformations, namely, the group of ∞^6 transformations* (8 *a*), *each of which carries each of the two reguli on the quadric into itself, and the set of ∞^6 transformations* (8 *b*), *each of which interchanges the two reguli.*

According to Theorem 11, there is a one-to-one correspondence between the transformations (8 *a*), (8 *b*) and the collineations in space which carry the quadric into itself.

THEOREM 13. *There are $2 \cdot \infty^6$ collineations of space which leave fixed a given nonsingular quadric surface.*

EXERCISES

1. Establish Theorems 2, 4 by use of § 9, Ex. 1.

2. Prove Theorem 9 by both the methods suggested in the text.

3. Show that there is a unique nonsingular quadric which contains two given skew lines and passes through three given points no two of which are coplanar with either of the given lines.

4. Count the number of quadrics in space, assuming that the general quadric is determined (*a*) as in Theorem 8, (*b*) as in Theorem 9, (*c*) as in the previous exercise.

5. How many lines are there which intersect each of four mutually skew lines? Discuss all possibilities.

6. Find the condition in terms of the *a*'s that the conic (7) be degenerate. Hence write the equation in plane coordinates of the quadric (1).

7. Characterize geometrically the involutory homologies and the skew involutions of space which leave a given nonsingular quadric in place.

8. Show that every nonsingular quadric is represented parametrically, in terms of homogeneous projective point coordinates on it, by equations of the form

$$\rho x_i = a_{i1}u_1v_1 + a_{i2}u_1v_2 + a_{i3}u_2v_1 + a_{i4}u_2v_2, \qquad (i = 1, 2, 3, 4), \quad |a_{ij}| \neq 0,$$

and conversely.

9. Deduce for the quadric

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = 0$$

the parametric representation

$$x_1 : i\,x_2 : x_3 : i\,x_4 = u_1v_2 + u_2v_1 : u_1v_2 - u_2v_1 : u_1v_1 - u_2v_2 : u_1v_1 + u_2v_2.$$

10. Dualize the contents of the text, starting with the equation in plane coordinates of the quadric (1).

11. Prove that a cross ratio of four rulings of a regulus on a quadric is equal to the corresponding cross ratio of the four points in which the rulings intersect a nondegenerate conic on the quadric. Hence show that the regulus meets two nondegenerate conics on the quadric in projective ranges of points.

12. What special property have the rulings on a sphere?

13. Show that a necessary and sufficient condition that a regulus lie on a paraboloid is that three and hence all of its finite rulings be parallel to a plane.

14. Show that the corresponding segments which finite rulings of one regulus of a paraboloid intercept on two finite rulings of the conjugate regulus are proportional.

12. Inversive Geometry on the Sphere. In this paragraph we propose to show, first, that the inversive geometry of the plane may be reproduced in all detail on the sphere and, secondly, that the resulting inversive geometry on the sphere is identical with the projective geometry on the sphere. It will follow that the inversive geometry of the plane is abstractly identical with the projective geometry on any nonsingular quadric which has a real trace but is without real rulings.

Stereographic Projection. The reproduction on the sphere of inversive geometry in the plane is affected by means of a transformation of the plane of inversion into the sphere which is known as stereographic projection. We shall first describe it as applying only to *real* points of the plane and sphere.

Think of the plane as the equatorial plane of the sphere and, from the north pole N of the sphere as center, project the plane on the sphere. To each finite point P of the plane corresponds a unique point P', other than N, of the sphere, and conversely. When P recedes indefinitely, P' approaches N as a limit. Accordingly, to N is ordered *the* point at infinity in the plane.

THEOREM 1 *a*. *Stereographic projection establishes between the real points of the plane of inversion and the real points of the sphere a continuous correspondence which is one-to-one without exception.*

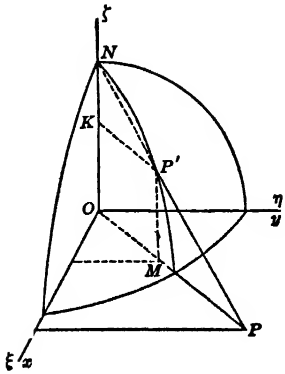


FIG. 7

Taking the radius of the sphere as the unit of length, introduce rectangular coordinates (x, y) and (ξ, η, ζ) in the plane and in space respectively, as indicated in Fig. 7. The equation of the sphere is

$$\xi^2 + \eta^2 + \zeta^2 = 1.$$

If P and P' have the coordinates (x, y) and (ξ, η, ζ), then, by similar triangles,

$$\frac{x}{\xi} = \frac{y}{\eta} = \frac{OP}{OM}, \qquad \frac{OP}{KP'} = \frac{1}{1 - \zeta},$$

so that

$$\frac{x}{\xi} = \frac{y}{\eta} = \frac{1}{1 - \zeta}.$$

Hence we obtain, as the equations of the transformation of (ξ, η, ζ) into (x, y):

$$(1\,a) \qquad x = \frac{\xi}{1 - \zeta}, \qquad y = \frac{\eta}{1 - \zeta}, \qquad x^2 + y^2 = \frac{1 + \zeta}{1 - \zeta}.$$

The equations of the inverse transformation are

$$(1\,b) \quad \xi = \frac{2\,x}{x^2 + y^2 + 1}, \qquad \eta = \frac{2\,y}{x^2 + y^2 + 1}, \qquad \zeta = \frac{x^2 + y^2 - 1}{x^2 + y^2 + 1}.$$

It is convenient to replace (ξ, η, ζ) by homogeneous coordinates (x_1, x_2, x_3, x_4) so chosen that the equation of the sphere takes the symmetric form

$$(2) \qquad x_1^2 + x_2^2 + x_3^2 + x_4^2 = 0.$$

This is accomplished by setting

$$\xi = \frac{x_1}{i\,x_4}, \qquad \eta = \frac{x_2}{i\,x_4}, \qquad \zeta = \frac{x_3}{i\,x_4}.$$

It is true that a real finite point does not possess real coordinates (x_1, x_2, x_3, x_4), but x_1, x_2, x_3 may always be taken as real, with x_4 as pure imaginary, so that the inconvenience is negligible.

The equations of the stereographic projection now become

$$
(3)\qquad
\begin{aligned}
\rho x_1 &= 2\,x, & x &= -\frac{x_1}{x_3 - i\,x_4},\\
\rho x_2 &= 2\,y, & y &= -\frac{x_2}{x_3 - i\,x_4},\\
\rho x_3 &= x^2 + y^2 - 1, & x^2 + y^2 &= -\frac{x_3 + i\,x_4}{x_3 - i\,x_4}.\\
\rho x_4 &= -\,i(x^2 + y^2 + 1), & &
\end{aligned}
$$

Stereographic Projection of the Complex Plane of Inversion. When (x, y) are replaced by $(u_1, u_2; v_1, v_2)$ where

$$u = x + i\,y, \qquad v = x - i\,y, \qquad \frac{u_1}{u_2} = u, \qquad \frac{v_1}{v_2} = v,$$

the first set of equations (3) becomes

$$(4)\qquad x_1 : i\,x_2 : x_3 : i\,x_4 = u_1v_2 + u_2v_1 : u_1v_2 - u_2v_1 : u_1v_1 - u_2v_2 : u_1v_1 + u_2v_2.$$

We recall that $(u_1, u_2; v_1, v_2)$ are homogeneous isotropic coordinates for the complex plane of inversion (Ch. XVIII, § 18). On the other hand, we know that equations (4) constitute a complete parametric representation of the complex points of the sphere in terms of $(u_1, u_2; v_1, v_2)$ as homogeneous projective coordinates on the sphere (§11, Ex. 9). Hence, equations (4) represent a transformation of the inversive plane into the sphere, which establishes between the complex points of the plane and the complex points of the sphere a perfect one-to-one correspondence. This transformation is the analytic extension, to *all* complex points of the inversive plane, of stereographic projection. Accordingly, we shall call it the stereographic projection of the complex inversive plane on the sphere. That it actually represents the geometric process of projection for all *finite* complex points of the plane is readily verified.*

THEOREM 1 *b*. *Stereographic projection establishes between the complex points of the inversive plane and the complex points of the sphere a continuous one-to-one correspondence.*

THEOREM 2. *The isotropic coordinates of a point in the plane and the projective coordinates of the corresponding point on the sphere are identical.*

* Since we cannot think of the complete complex plane of inversion, but only of its finite portion, as immersed in projective space, the process of stereographic projection cannot be applied to the infinite points of the plane. The fates of these points become, then, a matter of definition and they are fixed here by equations (4).

Properties of Stereographic Projection. Theorem 2 has a number of important consequences. In the first place, we recall that the equations

$$k_1u_1 + k_2u_2 = 0, \qquad l_1v_1 + l_2v_2 = 0$$

represent the two families of isotropic lines in the plane (Ch. XVIII, § 18). They also represent the two families of rulings on the sphere. But the rulings on the sphere are isotropic lines. Hence:

THEOREM 3. *Stereographic projection establishes a one-to-one correspondence between the isotropic lines in the plane and the isotropic lines on the sphere.*

In particular, to the isotropics at infinity in the plane correspond the isotropics through the north pole of the sphere.

THEOREM 4. *Stereographic projection establishes a one-to-one correspondence between the circles in the plane and the circles on the sphere.*

For, the circles in the plane are represented by the equations which are bilinear in u_1, u_2 and v_1, v_2 (Ch. XVIII, § 18), and these equations also represent the circles on the sphere, considered as the curves in which the planes of space meet the sphere (§ 11).

THEOREM 5. *Stereographic projection is conformal.*

The fact that the geometric process of stereographic projection carries the finite isotropic lines of the plane into isotropic lines on the sphere enables us to give a simple proof of the theorem. Let P be a finite point in the plane corresponding to a finite point P' on the sphere, and let C_1, C_2 be two curves in the plane intersecting in P and C_1', C_2' the corresponding curves through P' on the sphere. Denote the tangent lines to C_1, C_2 at P by T_1, T_2, and those of C_1', C_2' at P' by T_1', T_2'; the isotropic lines through P by I_1, I_2, and those through P' by I_1', I_2'. The lines T_1', T_2', I_1', I_2' lie in the tangent plane to the sphere at P', and the projection from N of this tangent plane on the plane of inversion evidently carries them into T_1, T_2, I_1, I_2. The cross ratios $(T_1'T_2',\ I_1'I_2')$, $(T_1T_2,\ I_1I_2)$ are then equal, and therefore, by Laguerre's definition of angle, the corresponding angles are equal.

We have achieved our first aim. Theorems 1–5 imply that stereographic projection carries the inversive geometry of the plane into precisely the same geometry on the sphere. Point goes into point, circle into circle, orthogonal circles into orthogonal circles, a pencil into a pencil (Ex. 3), and an orthogonal system into an orthogonal system.

Moreover, when inversion and circular transformations are defined on the sphere as in the plane, inversion goes into inversion and circular transformations into circular transformations.

On the sphere, all nondegenerate circles are obviously of one type; the distinction between proper circles and straight lines disappears completely. And, what is more important, all points are on the same footing; no points play special rôles. The truth of this we realized while still in the plane (Ch. XVIII, § 17, Th. 10, § 18), but it is brought out most strikingly on the sphere. The sphere is, in fact, the perfect example of the inversive domain.

THEOREM 6. *The circular transformations of the plane or the sphere are identical with the projective transformations of the sphere.*

For, equations (8 a), (8 b) of § 11 evidently represent the complex circular transformations of the plane or the sphere as well as the projective transformations of the sphere.

Herewith we have completed the proof of our second contention.

THEOREM 7. *The inversive geometry on the sphere is identical with the projective geometry on the sphere.*

The Inversive Geometry of the Sphere in its Relation to the Surrounding Projective Space. The transformation (3) carries the equation of a circle in the plane

$$(5) \qquad a_1'(x^2 + y^2) + a_2'x + a_3'y + a_4' = 0$$

into the equation of the plane of the corresponding circle on the sphere, namely

$$(6) \qquad a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4 = 0$$

where

$$(7) \quad a_1 = a_2', \qquad a_2 = a_3', \qquad a_3 = a_1' - a_4', \qquad a_4 = i(a_1' + a_4').$$

It is to be noted that, if the a''s are real, a_1, a_2, a_3 are real and a_4 is pure imaginary or zero.

If b' is a second circle in the plane and b the plane of the corresponding circle on the sphere, we readily establish, by means of (7) and the corresponding relations between the b's and b''s, the identities

$$(8) \qquad (a|b) \equiv (a', b'), \qquad (a|a) \equiv (a', a'),$$

where $(a|a)$, $(a|b)$ have their usual meanings and (a', a') (a', b') are the symbols of Ch. XVIII, §§ 1, 6.

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where $(a\,|\,a)$, $(a\,|\,b)$ have their usual meanings and (a', a') (a', b') are the symbols of Ch. XVIII, §§ 1, 6.

THEOREM 8. *Two circles on the sphere are orthogonal if and only if their planes are conjugate with respect to the sphere.*

For, by (8), the condition $(a', b') = 0$ that the two circles be orthogonal is equivalent to the condition $(a|b) = 0$ that the two planes be conjugate with respect to the sphere $(x|x) = 0$.

THEOREM 9. *A necessary and sufficient condition that two distinct points of the sphere be mutually inverse in a given circle on the sphere is that they be collinear with the pole of the plane of the circle.*

If P_1, P_2 are the given points, the line P_1P_2 is the line of intersection of the planes of two circles C_1, C_2 on the sphere which pass through P_1, P_2. If P_1, P_2 are mutually inverse in a given circle C on the sphere, C_1, C_2 are by definition orthogonal to C. The planes of C_1, C_2 are then conjugate to the plane m of C and so go through the pole M of this plane m. Hence their line of intersection P_1P_2 passes through M. Conversely, each two points P_1, P_2 of the sphere which are collinear with M are inverse in C, for the argument may be reversed.

Since the point M and the plane m are pole and polar, the points P_1, P_2 separate M and the intersection of their line with m harmonically. Hence, the inversion in the circle C is established by the involutory homology of space which has M as center and m as central plane.

THEOREM 10. *The inversions on the sphere are established by the involutory homologies of space which leave the sphere in place.*

THEOREM 11. *The Moebius involutions on the sphere are established by the skew involutions of space which carry the sphere into itself.*

The latter theorem is readily proved when it is recalled that a Moebius involution is the product of two inversions in mutually orthogonal circles (Ch. XVIII, § 17, Ex. 8).

LEMMA. *A necessary and sufficient condition that the linear transformation*

$$(9) \qquad x'_i = \sum_{j=1}^{4} a_{ij}x_j, \qquad (i = 1, 2, 3, 4), \quad \Delta = |a_{ij}| \neq 0,$$

has $(x|x)$ as an absolute invariant: $(x'|x') \equiv (x|x)$ is that its coefficients satisfy the relations

$$(10) \qquad \sum_{k=1}^{4} a_{ki}^2 = 1, \qquad \sum_{k=1}^{4} a_{ki}a_{kj} = 0, \quad (i \neq j,\ i, j = 1, 2, 3, 4).$$

We leave the proof to the reader. Let him also establish, as a

consequence of (10), the relation

$$(11) \qquad \Delta = \pm 1.$$

The transformations (9) for which the relations (10) are valid are known in algebra as the *quaternary orthogonal* substitutions or linear transformations.* Those for which $\Delta = 1$ are characterized as *direct*, and those for which $\Delta = -1$, as *indirect*. The direct transformations form a group, whereas the indirect ones do not.

The collineations (9) for which the relations (10) hold are the collineations which carry the sphere $(x|x) = 0$ into itself. Those which are direct carry each family of isotropics into itself and those which are indirect interchange the two families, as considerations of continuity readily show. Hence, we have, in accordance with Th. 6 and § 11, Th. 11:

THEOREM 12. *The direct (indirect) circular transformations on the sphere $(x|x) = 0$ are precisely the transformations established on the sphere by the direct (indirect) orthogonal collineations of space.*†

EXERCISES

1. What circles on the sphere correspond to the straight lines of the plane? What circle in the plane corresponds to the circle at infinity on the sphere?

2. Show analytically that the circles in the plane which correspond to the circles on the sphere whose planes are isotropic planes (§ 10, Exs. 13, 14) are the circles which are tangent to $x^2 + y^2 + 1 = 0$.

3. Prove that the circles of a pencil on the sphere, that is, the totality of circles corresponding to a pencil in the plane, lie in the planes of a pencil of planes. Classify real pencils of circles according to the relationships of the corresponding pencils of planes to the sphere.

4. What can you say about two pencils of planes which intersect the sphere in the circles of an orthogonal system?

5. Show that to the inversion in the unit circle of the plane corresponds the reflection of the sphere in the equatorial plane. What corresponds to the reflection of the plane in the x-axis? To the transformation $z' = 1/z$?

6. Prove Theorem 11.

* They are called "quaternary" because four variables are involved and "orthogonal" because $(x|x)$ is an absolute invariant. We have already had an example of ternary orthogonal substitutions in the equations of a rotation leaving fixed the origin of coordinates (§ 7).

† If only real transformations are desired, a_{i4}, a_{4i}, $i = 1, 2, 3$ should be taken as pure imaginary or zero and the remaining a's real. Then x_1', x_2', x_3' will be real and x_4' pure imaginary or zero, whenever x_1, x_2, x_3 are real and x_4 pure imaginary or zero.

7. What transformations of the plane correspond to the rotations of the sphere about the ζ-axis? To the rotations of the sphere about an arbitrarily chosen diameter?

8. Prove directly that the circular transformations on the sphere are precisely the transformations which are established by the collineations of space which leave the sphere in place.

9. Deduce from the theory of cross ratio on a quadric a theory of cross ratio in the *complex* plane of inversion. In particular, discuss cross ratio for four isotropics of the same family, for four points on an isotropic, and for four points on a nondegenerate circle (§ 11, Ex. 11). Show that two pairs of points on a nondegenerate circle are harmonic if and only if they are orthocyclic in the sense in which the term was defined for the real plane of inversion (Ch. XVIII, § 17).

10. Let P_1, P_2, P_3, P_4 be any four distinct points of the plane of inversion, and L_1, L_2, L_3, L_4 the four isotropics of the one family, M_1, M_2, M_3, M_4 the four isotropics of the other family, through them. Prove that, if P_1, P_2, P_3, P_4 are real points, the cross ratios (L_1L_2, L_3L_4), (M_1M_2, M_3M_4) are conjugate-complex, and that one of them is precisely the cross ratio (P_1P_2, P_3P_4) as defined in the real plane of inversion (Ch. XVIII, § 17).

11. If, in stereographic projection, the inversive plane is replaced by the projective plane, the process of projection may be applied to all the complex points of the plane. The resulting correspondence is, however, not one-to-one. Why?

12. Prove the Lemma and establish relation (11).

13. Show that relations (10) are equivalent to

$$\sum_{k=1}^{4} a_{ik}^2 = 1, \qquad \sum_{k=1}^{4} a_{ik}a_{jk} = 0, \qquad (i \neq j,\ i, j = 1, 2, 3, 4).$$

Suggestion. Begin by proving that the transformation $u_i = \sum_{j=1}^{4} a_{ji}u_j'$ $(i = 1, 2, 3, 4)$, carries $(u'|u') = 0$ into $(u|u) = 0$.

13. Tetracyclic Coordinates. In the preceding section we considered the ordered number sets (x_1, x_2, x_3, x_4) which satisfy the relation

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = 0 \tag{1}$$

as homogeneous coordinates of the points on the sphere. We shall now think of them as homogeneous coordinates of the corresponding points of the inversive plane. They are known, then, as *tetracyclic coordinates*, for reasons which we shall go into later.

Tetracyclic coordinates differ from previous pure * homogeneous coordinates for a two-dimensional manifold in that they number four, instead of the usual three, and must always satisfy the relation (1).

* That is, not mixed. The mixed isotropic coordinates $(u_1, u_2; v_1, v_2)$ are four in number, but only the *two* ratios $u_1 : u_2$, $v_1 : v_2$ count geometrically.

The analytic form which inversive geometry takes in terms of tetracyclic coordinates is evident from the last section. A circle is represented by a linear homogeneous equation

$$a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4 = 0, \tag{2}$$

whose coefficients are not all zero, and conversely. The equations of the group of direct circular transformations are those of the group of direct orthogonal substitutions

$$x_i' = \sum_{j=1}^{4} a_{ij}x_j, \qquad (i = 1, 2, 3, 4), \quad \Delta = 1, \tag{3}$$

where

$$\sum_{k=1}^{4} a_{ki}^2 = 1, \qquad \sum_{k=1}^{4} a_{ki}a_{kj} = 0, \quad (i \neq j, \ i, j = 1, 2, 3, 4).$$

The advantage of tetracyclic coordinates lies in the fact that equations (2) and (3) are linear. The quadratic element in the analytic theory has been concentrated in the relation (1).

Equations (4) of § 12 represent the transformation from isotropic coordinates to tetracyclic coordinates. The equations of the inverse transformation are

$$u_1 : u_2 = x_1 + i\,x_2 : -\,x_3 + i\,x_4 = x_3 + i\,x_4 : x_1 - i\,x_2,$$
$$v_1 : v_2 = x_1 - i\,x_2 : -\,x_3 + i\,x_4 = x_3 + i\,x_4 : x_1 + i\,x_2.$$

Circle Coordinates. There are only two types of complex circles in the inversive plane, the nondegenerate or proper circles (including the nonisotropic lines), and the degenerate or null circles; see Ch. XVIII, § 18.

THEOREM 1. *The circle* (2) *is proper or null according as* $(a|a) \neq 0$, *or* $(a|a) = 0$.

We shall call the ordered coefficients a_1, a_2, a_3, a_4 in equation (2) *homogeneous coordinates of the circle.* Evidently, each ordered quadruple (y_1, y_2, y_3, y_4) other than $(0, 0, 0, 0)$ constitutes coordinates of a circle. We shall speak of this circle as the circle y.

When the circle a is a null circle: $(a|a) = 0$, then (a_1, a_2, a_3, a_4) are also the coordinates of a point. This point is the center of the null circle. For, when $(a|a) = 0$, the plane $(a|x) = 0$ is the tangent plane to the sphere $(x|x) = 0$ at the point a.

THEOREM 2. *The tetracyclic coordinates of the center of a null circle are identical with the coordinates of the circle.*

We next note an obvious but important fact.

THEOREM 3. *The point r lies on the circle a if and only if* $(a|r) = 0$.

According to § 12, (8) and Ch. XVIII, § 10, (1), the angles θ between two proper circles a and b are given by

$$\cos\theta = \pm\frac{(a|b)}{\sqrt{a|a}\,\sqrt{b|b}}. \tag{4}$$

The condition for the orthogonality of any two circles is

$$(a|b) = 0, \tag{5}$$

and that for tangency,

$$(a|a)(b|b) - (a|b)^2 = 0. \tag{6}$$

Theorems 2 and 3, applied in conjunction with (5) and (6), inform us immediately that a null circle b is orthogonal or tangent to a given circle a if and only if its center b lies on the given circle.

A Metric Interpretation of Tetracyclic Coordinates. The transformation from rectangular to tetracyclic coordinates is, by § 12, (3),

$$\rho x_1 = 2x, \quad \rho x_2 = 2y, \quad \rho x_3 = x^2 + y^2 - 1, \quad \rho x_4 = \frac{1}{i}(x^2 + y^2 + 1).$$

THEOREM 4. *The tetracyclic coordinates of a finite point are proportional to relative powers of the point with respect to four mutually orthogonal proper circles.*

That each two of the four circles $x_i = 0$, $i = 1, 2, 3, 4$, are mutually orthogonal is readily verified. By a *relative power* of a finite point with respect to a proper circle, not a straight line, we mean the ratio of the actual power of the point to a determination of the radius.* The limit of this ratio, when the circle approaches a straight line as a limit, we take as the definition of a relative power of a point with respect to the line. It turns out to be twice a directed distance from the line to the point (Ex. 6).

The theorem is now clear. From it tetracyclic coordinates take their name.

* We agree here to recognize two radii, namely, the square roots of the "square of the radius." There are then two relative powers of a point with respect to a proper circle, just as there are two directed distances from a straight line to a point.

EXERCISES

1. For a real circle a, a_1, a_2, a_3 are taken as real and a_4 pure imaginary or zero. Show that the circle is, then, a circle with a real trace, a null circle, or a circle without a real trace according as $(a|a) > 0$, $= 0$, or < 0.

2. Prove that three circles belong to a pencil if and only if their homogeneous coordinates are linearly dependent.

3. Show that the inverse of the point x in the proper circle a is the point

$$x' = 2\,(a|x)\,a - (a|a)\,x.$$

4. Prove that the formula of Ex. 3 also represents the inverse of the circle x in the proper circle a.

5. Three circles are given, no one of which belongs to the pencil conjugate to that determined by the other two. Show that the three circles, each of which is coaxal with two of the given circles and orthogonal to the third, are coaxal.

6. Show that a relative power of a finite point with respect to a straight line is twice a directed distance from the line to the point.

7. Prove that a relative power of a finite point x with respect to the proper circle a is of the form

$$\pm\, k\,\frac{(a|x)}{\sqrt{a|a}},$$

where k depends only on x, not on a.

8. Show that four distinct real circles, if they are mutually orthogonal, must all be proper, and that three must have real traces and the fourth no real trace.

9. The coordinates (x_1', x_2', x_3', x_4') defined by equations (3), considered as a change of coordinates, are known as *general tetracyclic coordinates*.* Show that inversive geometry has the same analytic form in terms of them as in terms of the original tetracyclic coordinates.

10. Prove that Theorem 4 is true for the general tetracyclic coordinates of Ex. 9.

Suggestion. Employ Ex. 7 and § 12, Ex. 13.

14. Line Geometry. Pluecker Line Coordinates. We bring the book to a close with a brief introduction to the line geometry of space. Since space contains ∞^4 lines, this geometry is four-dimensional. In it points and planes, that is, sheaves of lines and planes of lines, are two-dimensional, and pencils of lines and reguli, one-dimensional.

Coordinates for a line in space may be obtained by considering the line as determined by two points or as determined by two planes. Dual line coordinates arrived at in these two ways were developed by

* The most general tetracyclic coordinates are those defined by the general linear transformation of x_1, x_2, x_3, x_4. In distinction to them, the coordinates of the text and exercise are often called *orthogonal* tetracyclic coordinates.

the founder of line geometry, Julius Pluecker, and were called by him *ray coordinates* and *axis coordinates*, respectively. Both sets of coordinates are homogeneous, are six in number, and satisfy a quadratic relation. They resemble then, in structure, the tetracyclic coordinates in the inversive plane.

Ray Coordinates. Let a line L be given and choose on it two distinct points x, y. From the matrix of the coordinates (x_1, x_2, x_3, x_4), (y_1, y_2, y_3, y_4) of the two points form the two-rowed determinants

$$p_{ij} = x_i y_j - x_j y_i, \quad (i \neq j,\ i, j = 1, 2, 3, 4). \tag{1}$$

These twelve determinants are negatives of one another in pairs:

$$p_{ji} = -p_{ij}.$$

Accordingly, we restrict ourselves to six of them:

$$p_{12},\ p_{13},\ p_{14},\ p_{34},\ p_{42},\ p_{23}. \tag{2}$$

These six p's are *Pluecker ray coordinates* of the line L. They satisfy the relation

$$p_{12}p_{34} + p_{13}p_{42} + p_{14}p_{23} = 0, \tag{3}$$

as is readily proved by expanding the determinant in the identity $|x\,y\,x\,y| = 0$.*

If $x' = k_1x + l_1y$, $y' = k_2x + l_2y$ are any two distinct points on L,

$$p'_{ij} = x'_i y'_j - x'_j y'_i = (k_1l_2 - k_2l_1)p_{ij}.$$

Hence, each two sets of ray coordinates of L are proportional.

If L does not intersect the edge A_3A_4 of the tetrahedron of reference, $p_{12} \neq 0$. For, when L is thought of as determined by the distinct points $P_2 : (x_1, 0, x_3, x_4)$, $P_1 : (0, y_2, y_3, y_4)$ in which it meets the faces $x_2 = 0$, $x_1 = 0$ which intersect in A_3A_4, its coordinates are

$$p_{12} = \begin{cases} y_2x_1, & p_{13} = y_3x_1, \quad p_{14} = y_4x_1, \\ \quad\cdot & \qquad\qquad\qquad\qquad\qquad p_{34} = x_3y_4 - x_4y_3, \\ x_1y_2, & p_{23} = -x_3y_2,\ p_{42} = x_4y_2, \end{cases} \tag{4}$$

and, since $x_1y_2 \neq 0$, $p_{12} \neq 0$.

We are now in a position to give a simple proof that six numbers (2) which satisfy (3) and are not all zero are the coordinates of a line. We may assume, for example, that $p_{12} \neq 0$. Then the six equations (4)

* It is to be noted that the number of inversions in each of the orders, 1 2 3 4, 1 3 4 2, 1 4 2 3, in the subscripts of the terms of (3) is even.

The identity $|x\,y\,x\,y| = 0$ is most easily expanded into (3) by means of Laplace's rule; see Bôcher, *Higher Algebra*, Ch. II, § 8.

in $y_2, y_3, y_4, x_1, x_3, x_4$ are compatible and determine two points P_1, P_2 whose line has the given numbers as its coordinates. Moreover, since coordinates of the two points are obviously

$$(5) \qquad P_1 : (0, p_{12}, p_{13}, p_{14}), \qquad P_2 : (p_{12}, 0, -p_{23}, p_{42}), \qquad p_{12} \neq 0,$$

the line does not intersect A_3A_4.

Axis Coordinates. If u, v are two distinct planes through the line L, the six numbers

$$(6) \qquad q_{12}, q_{13}, q_{14}, q_{34}, q_{42}, q_{23},$$

where

$$q_{ij} = u_iv_j - u_jv_i, \qquad (i, j = 1, 2, 3, 4),$$

are *Pluecker axis coordinates* of L. They are connected by the relation

$$(7) \qquad q_{12}q_{34} + q_{13}q_{42} + q_{14}q_{23} = 0.$$

It is evident that the justification of the definition is the exact dual of that in the case of ray coordinates.

The Relations between Ray and Axis Coordinates. Once more we think of the line L as not intersecting $A_3A_4 : p_{12} \neq 0$. Then coordinates of two distinct points on L, expressed in terms of the ray coordinates of L, are given by (5), and coordinates of the two planes through L which contain respectively A_3 and A_4 are readily found to be

$$(p_{42}, p_{14}, 0, -p_{12}), \qquad (p_{23}, -p_{13}, p_{12}, 0).$$

The axis coordinates q of L, determined by these planes, are

$$(8) \qquad q_{ij} = \lambda p_{kl}, \qquad \lambda = p_{12} \neq 0,$$

where $i\,j$ ranges over the combinations 12, 13, 14, 34, 42, 23 and $k\,l$ over the *complementary* combinations 34, 42, 23, 12, 13, 14.

THEOREM 1. *The ray coordinates p and the axis coordinates q of a line are connected by the equations*

$$p_{12} : p_{13} : p_{14} : p_{34} : p_{42} : p_{23} = q_{34} : q_{42} : q_{23} : q_{12} : q_{13} : q_{14}.$$

The content of the theorem is frequently expressed in brief by saying that the coordinates of the one kind are proportional to the complementary coordinates of the other kind.

Fundamentals of Line Geometry in Line Coordinates. The two lines with ray coordinates p and p' determined respectively by the pairs of points x, y and x', y' intersect if and only if $|x\,y\,x'y'| = 0$ or, as is

readily shown by expanding the determinant, if and only if

$$p_{12}p'_{34} + p_{13}p'_{42} + p_{14}p'_{23} + p_{34}p'_{12} + p_{42}p'_{13} + p_{23}p'_{14} = 0.$$

THEOREM 2. *A necessary and sufficient condition that the two lines with ray coordinates p, p' intersect is that*

$$(p, p') = 0,$$

where

(9) $\quad (p, p') \equiv p_{12}p'_{34} + p_{13}p'_{42} + p_{14}p'_{23} + p_{34}p'_{12} + p_{42}p'_{13} + p_{23}p'_{14}.$

It follows, by Th. 1, that, if q, q' are axis coordinates of the two lines, the condition for intersection is $(q, q') = 0$.

We shall make frequent use of the symbol (p, p') defined by (9), and also of the related symbol

(10) $\quad (p, p) = 2\,(p_{12}p_{34} + p_{13}p_{42} + p_{14}p_{23}),$

whose vanishing is the condition that the six p's, not all zero, be the coordinates of a line.

A finite number of lines shall be said to be linearly dependent if their ray coordinates are linearly dependent or, what is the same thing, if their axis coordinates are linearly dependent.

The theory of linear dependence and linear combination is affected by the fact that the coordinates of a line must always satisfy the relation $(p, p) = 0$ or $(q, q) = 0$. Let us consider, for example, the lines linearly dependent on two distinct lines p', p''. We write as usual

(11) $\quad p = k_1p' + k_2p''.$

The p's thus defined are the coordinates of a line when and only when k_1, k_2 satisfy the equation

$$(k_1p' + k_2p'', k_1p' + k_2p'') = 0,$$

or

$$k_1^2(p', p') + 2\,k_1k_2(p', p'') + k_2^2(p'', p'') = 0.$$

Since $(p', p') = 0$ and $(p'', p'') = 0$, this equation reduces to

(12) $\quad k_1k_2(p', p'') = 0.$

If $(p', p'') \neq 0$, (12) is satisfied only if $k_1 = 0$ or $k_2 = 0$. Hence, *the only lines linearly dependent on two skew lines are the lines themselves.*

If $(p', p'') = 0$, (12) is an identity and every linear combination (11) represents a line. In this case, we have:

THEOREM 3. *The lines which are linearly dependent on two distinct intersecting lines are the lines of the pencil determined by them.*

For, think of p', p'' as determined by the point x common to the given lines and two points y', y'' lying respectively on them. Then

$$p_{ij} = k_1 \begin{vmatrix} x_i & x_j \\ y'_i & y'_j \end{vmatrix} + k_2 \begin{vmatrix} x_i & x_j \\ y''_i & y''_j \end{vmatrix} = \begin{vmatrix} x_i & x_j \\ k_1y'_i + k_2y''_i & k_1y'_j + k_2y''_j \end{vmatrix}.$$

Thus, p is the line joining x to the general point $k_1y' + k_2y''$ of the range determined by y', y''.

THEOREM 4. *Three distinct lines are linearly dependent if and only if they belong to the same pencil.*

If the word "distinct" were omitted, the theorem would be false. Why?

EXERCISES

1. Determine the ray coordinates of the edges of the tetrahedron of reference.

2. Characterize the ray coordinates of a line which (*a*) meets the edge A_1A_2 of the tetrahedron of reference; (*b*) meets the opposite edges A_1A_2, A_3A_4; (*c*) lies in the face a_1; (*d*) goes through the vertex A_1.

3. Describe the lines whose ray coordinates satisfy the equations:
(*a*) $p_{23} = 0$; (*b*) $p_{23} = 0$, $p_{14} = 0$; (*c*) $p_{23} = 0$, $p_{34} = 0$;
(*d*) $p_{23} = 0$, $p_{34} = 0$, $p_{42} = 0$; (*e*) $p_{23} = 0$, $p_{42} = 0$, $p_{12} = 0$.

4. Justify the definition of the q's as line coordinates.

5. Write a condition necessary and sufficient that the line with ray coordinates p meet the line with axis coordinates q.

6. Prove Theorem 3, using axis coordinates. Establish Theorem 4.

7. Three linearly independent lines go through a point. Prove that the lines which are linearly dependent on them are the lines of the sheaf determined by them.

8. State and prove the dual of the theorem of the previous exercise.

9. Prove that four lines, no three linearly dependent, are linearly dependent if and only if every line which meets three of them intersects the fourth. Show that then the four lines (*a*) belong to a sheaf or to a plane of lines, or (*b*) intersect in pairs so that the pencils determined by the two pairs have a line in common but lie in different planes and have different vertices, or (*c*) belong to a regulus.

10. Show that there are two lines, distinct or coincident, which intersect four given linearly independent lines, or all the lines of a pencil intersect the four lines. When does the latter case occur?

15. Linear Complexes. Null Systems. A three-parameter family of lines is known as a *complex of lines*. Since the number of lines in space is ∞^4, a single equation homogeneous in the ray or the axis co-

ordinates of a line, but not equivalent to $(p, p) = 0$, represents a complex. When the equation is linear, the complex is called a *linear complex.*

The two equations in ray and axis coordinates

$$(1) \qquad \sum a_{ij}p_{ij} = 0, \qquad \sum b_{ij}q_{ij} = 0,$$

where $i\,j$ ranges over the combinations 12, 13, 14, 34, 42, 23, represent the same linear complex, provided that

$$b_{ij} = \lambda\, a_{kl} \quad (\lambda \neq 0) \qquad \text{or} \qquad a_{ij} = \mu\, b_{kl} \quad (\mu \neq 0),$$

when $i\,j$ ranges over the given, and $k\,l$ over the complementary, combinations.

It is clear, from the relationship between the a's and the b's, that (a, a) and (b, b) vanish simultaneously. In this case, the a's and b's may be thought of respectively as the axis and ray coordinates of the same line. The complex consists, then, of all the lines which intersect this line, including the line itself. The line is known as the *axis* of the complex, and the complex is said to be *special.*

We turn to the general case:

$$(a, a)\,(b, b) \neq 0,$$

and rewrite equations (1) in the forms

$$(2) \qquad \sum a_{ij}(x_iy_j - x_jy_i) = 0, \qquad \sum b_{ij}(u_iv_j - u_jv_i) = 0.$$

Introducing new a's and b's:

$$a_{ji} = -\,a_{ij}, \qquad a_{kk} = 0, \qquad b_{ji} = -\,b_{ij}, \qquad b_{kk} = 0,$$

where $j\,i$ ranges over the combinations 21, 31, 41, 43, 24, 32 and $k = 1, 2, 3, 4$, we may replace equations (2) by

$$(3) \qquad \sum_{i,j}^{1-4} a_{ij}x_iy_j = 0, \quad a_{ji} = -\,a_{ij}, \qquad \sum_{i,j}^{1-4} b_{ij}u_iv_j = 0, \quad b_{ji} = -\,b_{ij},$$

where now i and j range independently over 1, 2, 3, 4 in the usual fashion.

The determinants $|a_{ij}|$ and $|b_{ij}|$ resulting from the accession of the new a's and b's are evidently skew-symmetric. Their values are readily found to be:

$$|a_{ij}| = (a, a)^2 \neq 0, \qquad |b_{ij}| = (b, b)^2 \neq 0.$$

For a given point y the first of equations (2) or (3) represents the locus of a point x which moves so that the line joining it to the point y is always a line of the complex. Since the equation is linear in the

x's with coefficients which cannot all vanish, the locus is a plane. Hence, the lines of the complex which go through a given point form a pencil.

Similarly, the lines of the complex which lie in a given plane form a pencil. For, the envelope of a plane u which moves so that the line in which it intersects a given plane is always a line of the complex is the point which is represented by the second of equations (2) or (3), when the v's are coordinates of the given plane.

THEOREM 1. *A nonspecial linear complex has a pencil of lines in every plane and a pencil of lines through every point in space.*

The complex orders to each point a unique plane, the plane of the pencil of lines at the point, and to each plane a unique point, the vertex of the pencil of lines in the plane. Moreover, if it orders to the point P the plane p, it evidently orders to the plane p the point P.

The point-plane involution of space thus established is represented by either of the equations (3), since the first of these equations defines the plane p corresponding to a given point P, and the second determines the point P corresponding to a given plane p.

Thus, equations (3) represent the same involutory correlation of space. Since $a_{12}, a_{13}, a_{14}, a_{34}, a_{42}, a_{23}$ are subject only to the condition $(a, a) \neq 0$, the sixteen a_{ij}'s are subject only to the condition that $|a_{ij}|$ be skew-symmetric, $\neq 0$. Hence, to each nonspecial complex (1) corresponds an involutory correlation (3), and conversely.

An involutory correlation with skew-symmetric determinant is known as a *null system.* We may, then, state our result as follows:

THEOREM 2. *A nonspecial linear complex determines a null system, and conversely.*

It is customary to employ the same terminology for a null system as for a polar system. The general properties of the null system, that is, those depending only on the fact that the system is an involutory correlation, are identical with those of a polar system (§ 9). Of the special properties of the null system, bearing on its relationship to the corresponding linear complex, we already have

THEOREM 3. *A point and a plane are pole and polar in the null system if and only if the point is the vertex of the pencil of lines of the complex which lies in the plane, or the plane is the plane of the pencil of lines of the complex which issues from the point.*

THEOREM 4. *Every point lies in its polar plane and every plane contains its pole; in other words, every point and every plane is self-conjugate.*

Since the polar plane of a point contains every line of the complex passing through the point, the polar plane p of a point P on a line L of the complex always contains L. Consequently, as P traces L, p rotates about $L : L$ is its own polar line.

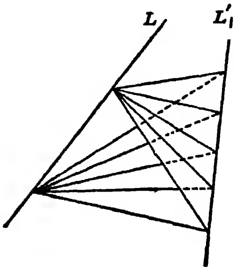

FIG. 8

If L is not a line of the complex, the polar plane p of a point P on L never contains L. As P traces L, p rotates about L', the polar line of L. Since the lines of the complex which meet L are precisely the lines through the points P lying in the planes p, they all meet L' and clearly exhaust all the lines which meet both L and L' (Fig. 8).

THEOREM 5. *The lines of the complex are self-polar. Two lines not belonging to the complex are mutually polar if and only if the lines of the complex which meet the one intersect the other, or if and only if the lines which meet both all belong to the complex.*

Incidentally, we have determined all the lines of the complex which intersect a given line.

THEOREM 6. *There are ∞^2 lines of the complex intersecting a given line. If the line is not in the complex, the ∞^2 lines are all the lines which meet it and its polar line. If the line belongs to the complex, the ∞^2 lines are distributed with respect to it as are the tangent lines to a nonsingular quadric surface at the points of a ruling.*

We leave to the reader the justification of the description in the latter case (Ex. 2).

Since two polar lines not belonging to the complex are skew lines, a line meets its polar if and only if it belongs to the complex and so is its own polar. Hence:

THEOREM 7. *The self-conjugate lines of the null system coincide with the self-polar lines, in the lines of the complex.*

EXERCISES

1. Prove directly that equations (3) represent the same involutory correlation.

2. Justify the description, in Theorem 6, of the lines of the complex which meet a given line of the complex; see § 9, Ex. 1.

3. A tetrahedron whose vertices and faces are pole and polar with respect to a null system is said to be self-polar. Show that a necessary and sufficient condition that a tetrahedron be self-polar is that two opposite edges be polar lines or that two pairs of opposite edges belong to the complex.

4. Show that four mutually skew lines which are polar in pairs in a null system belong to a regulus.

5. Prove that the lines of a nonspecial complex which intersect two nonconjugate lines form a regulus.

6. Show that the polar of a regulus with respect to a null system is a regulus. When is a regulus self-polar?

7. Prove that five lines are linearly independent if and only if they are intersected by at most one (singly counting) line.

8. Show that there is a unique linear complex containing five given lines which are intersected by at most one line and that the complex is nonspecial if no line meets the five given lines.

9. Prove that there is a unique null system with respect to which two pairs of lines of a regulus are polar lines.

10. Show that there is a unique null system in which two given lines are mutually polar and a third self-polar, provided the three lines are skew to one another.

16. Continuation. Metric Properties. In order to obtain a clearer picture of a linear complex, considered as a whole, we proceed to discuss it with reference to the metric geometry of Cartesian space. We assume that the complex is nonspecial and real.

If L is a finite line through the pole A_3 of the plane at infinity, the polar of L is a line L' in the plane at infinity and the planes determined by L' and the finite points of L, that is, the polar planes of the finite points of L, are all parallel. In particular, if L' is the polar line of A_3 with respect to the absolute conic, the polar planes of the points of L are all perpendicular to L, and conversely.

THEOREM 1. *There is a unique finite line which is perpendicular to the polar planes of the finite points lying on it.*

This line is called the *axis* of the complex.

If A_4 is an arbitrarily chosen finite point on the axis L, and A_1, A_2 are any two points on L' which are conjugate with respect to the absolute, the finite edges of the tetrahedron $A_1A_2A_3A_4$ are mutually perpendicular, and the tetrahedron may be used as tetrahedron of reference for rectangular Cartesian coordinates. Since A_1A_2, A_3A_4 are polar lines, the remaining edges of the tetrahedron belong to the

complex and the equation of the complex in ray coordinates becomes

$$(1) \qquad a_{12}p_{12} + a_{34}p_{34} = 0, \qquad a_{12}a_{34} \neq 0.$$

Restricting ourselves to the finite domain and thinking of the line p as determined by the points with nonhomogeneous Cartesian coordinates (x, y, z), $(\overline{x}, \overline{y}, \overline{z})$, we replace (1) by

$$(2) \qquad x\,\overline{y} - \overline{x}\,y + k(z - \overline{z}) = 0, \qquad k \neq 0.$$

Since the origin A_4 was any finite point on L and the points at infinity A_1 and A_2 in the directions of the axes x and y were any two points on L' conjugate with respect to the absolute, it is reasonable to expect that equation (2) is invariant with respect to any screw motion about the axis of z—the axis of the complex. That this is, indeed, the case may be readily verified analytically.

THEOREM 2. *A nonspecial linear complex is invariant with respect to the group of screw motions about its axis.*

It is clear from the definition of the axis that the lines which intersect the axis at right angles belong to the complex. To visualize the complex, we seek a picture of the lines of the complex intersecting an arbitrary ray R which issues from a point of the axis and is perpendicular to it. Since, by Th. 2, this picture is the same for every ray R, it suffices to construct it for one ray, for example, the positive half of the x-axis.

The equation of the polar plane p of the point P: $(r, 0, 0)$, $r \geqq 0$, on the positive half of the axis of x, obtained by setting $\overline{x} = r$, $\overline{y} = \overline{z} = 0$ in (2), is

$$(3) \qquad r\,y - k\,z = 0.$$

Evidently p goes through the axis of x and meets the (y, z)-plane in the line through the origin whose slope with respect to the y-axis is r/k. Hence, p may be obtained by rotating the (x, y)-plane about the axis of x through the angle θ defined by

$$(4) \qquad \tan\theta = \frac{r}{k},$$

where the positive direction of rotation is that from the positive y-axis to the positive z-axis.

The desired picture is now clear. As the point P, starting from the origin, recedes indefinitely on the positive half of the x-axis, the plane p, starting from the horizontal position, rotates gradually about the

x-axis and approaches the vertical position as a limit. If the rotation of p is viewed from a point on the negative axis of x, the direction of rotation is that of a right-handed or left-handed screw according as $k > 0$ or < 0.

EXERCISES

1. Prove Theorem 2.

2. Prove that there is a two parameter family of helices, or circular screws, which have the z-axis as axis, rise a given vertical distance with each turn about the axis, and wind about the axis in the same direction. Prove that, if $2\pi d$ is the given distance, the equations

$$x = \rho \cos(\phi + \alpha), \quad y = \rho \sin(\phi + \alpha), \quad z = d\,\phi, \qquad \rho \geqq 0,$$

constitute a representation, in terms of the parameter ϕ, of the helix of the family which passes through the point in the (x, y)-plane with polar coordinates (ρ, α).

3. Show that there is a family of helices of the type just described which is associated with a given nonspecial linear complex in such a way that the polar plane of an arbitrary point P is the normal plane at P of the helix which passes through P.

17. Linear Congruences. Reguli. A two-parameter family of lines is known as a *congruence of lines*. Two independent equations homogeneous in ray or axis coordinates determine, in general, a congruence. If the equations are both linear, the congruence is called *linear*.

We shall represent a linear congruence by two linearly independent linear equations in the ray coordinates p, writing the equations in the symbolic forms

$$(1) \qquad (a, p) = 0, \qquad (b, p) = 0.$$

The lines of the congruence are the lines common to the two linear complexes represented by the two equations taken individually. Hence, they belong also to every complex of the *pencil of linear complexes:*

$$(2) \qquad k\,(a, p) + l\,(b, p) = 0 \qquad \text{or} \qquad (k\,a + l\,b, p) = 0,$$

and are the lines common to each two distinct complexes of this pencil.

Theorem 1. *The lines of a linear congruence belong to each of the complexes of a pencil of linear complexes associated with the congruence, and are precisely the lines common to each two complexes of the pencil.*

Since the lines of the congruence belong to whatever special complexes are contained in the pencil, they must intersect the axes of the special complexes of the pencil. These axes are called the *directrices* of the congruence.

THEOREM 2. *The lines of a linear congruence intersect every directrix of the congruence.*

The general complex of the pencil (2) is special when and only when k, l satisfy the equation

$$k^2(a, a) + 2\,kl(a, b) + l^2(b, b) = 0. \tag{3}$$

Hence, the congruence has two distinct directrices, one doubly counting directrix, or ∞^1 directrices. It is known respectively in these three cases as *nonparabolic*, *parabolic*, and *special*.

For a *nonparabolic congruence*, the pencil contains two distinct special complexes. If these are taken as the complexes a, b, then $(a, a) = 0$, $(b, b) = 0$ and hence, since (3) may not reduce to the identity, $(a, b) \neq 0$. Consequently, the two directrices of the congruence are skew lines, and the congruence consists of all the lines which intersect them both.

THEOREM 3. *A linear congruence of lines consists, in general, of the lines which meet two skew lines.*

Since the lines of the congruence belong to every complex of the pencil, we have, by § 15, Th. 5:

THEOREM 4. *The directrices of the general linear congruence are polar lines with respect to every nonspecial complex of the associated pencil of linear complexes.*

A *parabolic congruence* has a single (doubly counting) directrix L and consists of the lines of a nonspecial complex of the pencil which meet L. If L did not belong to this nonspecial complex, these lines would intersect a second line skew to L, by § 15, Th. 6, and the congruence would be nonparabolic. Hence, L is a line of the nonspecial complex and the lines of the congruence are distributed with respect to it as are the tangent lines to a nonsingular quadric at the points of a ruling.

The ∞^1 directrices of a *special congruence* are the lines of a pencil. The congruence consists of all the lines through the vertex of the pencil and all the lines in the plane of the pencil.

Reguli. We define a regulus anew as the totality of lines which satisfy three linearly independent linear equations in, let us say, the ray coordinates p:

$$(a^{(1)}, p) = 0, \qquad (a^{(2)}, p) = 0, \qquad (a^{(3)}, p) = 0. \tag{4}$$

The regulus consists, then, of the lines common to the three linearly

independent linear complexes represented by the individual equations (4), or of the lines common to any three linearly independent complexes of the *sheaf of complexes*

$$(5) \qquad k_1(a^{(1)}, p) + k_2(a^{(2)}, p) + k_3(a^{(3)}, p) = 0.$$

THEOREM 5. *The lines of a regulus belong to each of the complexes of a sheaf of linear complexes associated with the regulus, and are precisely the lines common to any three linearly independent complexes of the sheaf.*

The axes of the special complexes of the sheaf are called *directrices* of the regulus.

THEOREM 6. *The lines of a regulus intersect every directrix of the regulus.*

The condition that the general complex (5) of the sheaf be special is that k_1, k_2, k_3 satisfy the quadratic equation

$$(6) \qquad \sum_{i,j}^{1-3}(a^{(i)}, a^{(j)})k_ik_j = 0.$$

Let us interpret k_1, k_2, k_3 as homogeneous point coordinates in a projective plane, and think of the point (k_1, k_2, k_3) of the plane as ordered to the complex (5) of the sheaf, and vice versa. There is thus established a one-to-one correspondence between the points of the plane and the complexes of the sheaf which has the following properties: (*a*) to three noncollinear points of the plane correspond three linearly independent complexes; (*b*) to a point of the point conic (6) corresponds a special complex and hence a directrix of the regulus; (*c*) to two conjugate points of the point conic (6) correspond two intersecting directrices of the regulus. Each of these properties is reversible. Only the third requires proof, and the proof of it is straightforward.

Nondegenerate Reguli. If the point conic (6) is nondegenerate, the regulus is said to be nondegenerate. Since in this case two distinct points of the conic cannot be conjugate, each two directrices of the regulus are skew lines. Hence, the regulus consists of the ∞^1 lines which meet three skew lines (Ths. 5, 6). It is then a regulus in the sense in which the term was originally defined; see §§ 8, 11.

THEOREM 7. *The directrices of a nondegenerate regulus constitute the conjugate regulus.*

Since the lines of the regulus meet every directrix, the ∞^1 directrices belong to the conjugate regulus. Conversely, a line p^* of the con-

jugate regulus is a directrix of the given regulus. For, if $p^{(1)}$, $p^{(2)}$, $p^{(3)}$ are three lines of the given regulus and $a^{(1)}$, $a^{(2)}$, $a^{(3)}$ three directrices, p^* is a solution of the three linear homogeneous equations

$$(p^{(1)}, p^*) = 0, \qquad (p^{(2)}, p^*) = 0, \qquad (p^{(3)}, p^*) = 0,$$

of which $a^{(1)}$, $a^{(2)}$, $a^{(3)}$ are three linearly independent solutions; hence p^* is a linear combination of the three directrices $a^{(1)}$, $a^{(2)}$, $a^{(3)}$, and so is itself a directrix.

Degenerate Reguli. These are of three different types, according as the rank r of the matrix of the quadratic form in (6) is two, one, or zero. We content ourselves with a statement of the results in each case.

A: $r = 2$. The regulus consists of two pencils of lines which have different vertices and planes, but a common line. Its directrices constitute a second regulus of the same type.

B: $r = 1$. The regulus is a single doubly counting pencil of lines, and the regulus of directrices coincides with it.

C: $r = 0$. In this case, the regulus consists of ∞^2 lines, namely all the lines of a sheaf of lines or all the lines of a plane of lines. Here, too, the regulus of directrices coincides with the given regulus.

We may now draw the following conclusion.

THEOREM 8. *The directrices of a given regulus constitute a regulus of the same type as the given regulus.*

EXERCISES

1. Show that there is a unique linear congruence which contains four given linearly independent lines.

2. Prove that a parabolic congruence actually consists of the lines tangent to a nonsingular quadric at the points of a ruling.

3. Show that a special congruence consists of the lines which meet two intersecting lines.

4. Prove that all the lines of a congruence meeting a line which does not belong to the congruence and is not a directrix of the congruence form a regulus.

5. Show that there is a unique regulus which contains three given linearly independent lines.

6. Prove that, if a linear complex does not contain a linear congruence, the two have a regulus in common.

7. Establish the facts concerning degenerate reguli.

8. Find the number of linear complexes, linear congruences, and reguli in space.

9. Show that the lines linearly dependent on five linearly independent lines are the lines of a linear complex, and conversely.

10. Show that the lines linearly dependent on four linearly independent lines are the lines of a linear congruence, and conversely.

11. The lines linearly dependent on three linearly independent lines are the lines of a regulus (§ 14, Ex. 9). Is the converse true?

12. Why is it that a regulus, which has in general ∞^1 lines, may have ∞^2 lines, whereas a linear congruence always has ∞^2 lines, and a linear complex always has ∞^3 lines?

13. Show that every linear complex which contains a given linear congruence belongs to the pencil of linear complexes associated with the given congruence.

14. Does every linear complex which contains a given regulus belong to the sheaf of linear complexes associated with the regulus? If your answer is in the negative, justify it by an example.

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